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A 3D spinorial view of the exceptional root systems

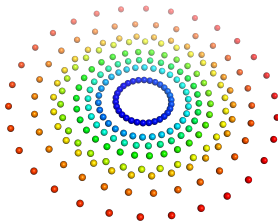
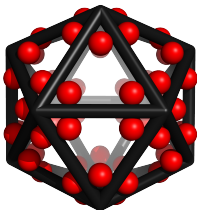
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Integrable Systems at Newcastle – February 6, 2016

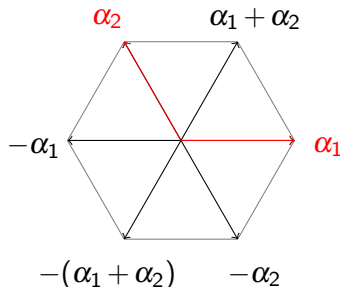
Main results

- Each 3D root system induces a 4D root system
- H_3 (icosahedral symmetry) induces the E_8 root system
- Clifford algebra is a very natural framework for root systems and reflection groups



- 1 Root systems and Clifford algebras
 - Root systems
 - Clifford Basics
- 2 H_4 as a rotation group I: 3D to 4D spinor induction, Trinities and McKay correspondence
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Root systems



reflection/Coxeter groups

Root system Φ : set of vectors α in a **vector space** with an **inner product** such that

1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$

2. $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

Simple roots: express every element of Φ via a **\mathbb{Z} -linear combination**.

$$s_\alpha : v \rightarrow s_\alpha(v) = v - 2 \frac{(v|\alpha)}{(\alpha|\alpha)} \alpha$$

Cartan Matrices

Cartan matrix of α_i s is
$$A_{ij} = 2 \frac{(\alpha_i | \alpha_j)}{(\alpha_i | \alpha_i)} = 2 \frac{|\alpha_j|}{|\alpha_i|} \cos \theta_{ij}$$

$$A_2: A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal, simple link = roots at $\frac{\pi}{3}$, link with label m = angle $\frac{\pi}{m}$.

$$A_3 \circ - \circ - \circ$$

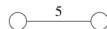
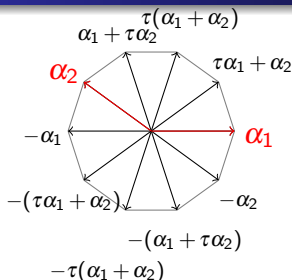
$$B_3 \circ - \overset{4}{\circ} - \circ$$

$$H_3 \circ - \overset{5}{\circ} - \circ$$

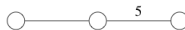
$$I_2(n) \circ - \overset{n}{\circ}$$

Lie groups to Lie algebras to Coxeter groups to root systems

- **Lie group**: manifold of continuous symmetries (gauge theories, spacetime)
- **Lie algebra**: infinitesimal version near the identity
- Non-trivial part is given by a root lattice
- **Weyl** group is a **crystallographic** Coxeter group:
 $A_n, B_n/C_n, D_n, G_2, F_4, E_6, E_7, E_8$ generated by a **root system**.
- So via this route root systems are always crystallographic.
Neglect non-crystallographic root systems $I_2(n), H_3, H_4$.

Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$ 

$$A = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate **rotational symmetries of order 5**

linear combinations now in the **extended integer ring**

$$\mathbb{Z}[\tau] = \{a + \tau b \mid a, b \in \mathbb{Z}\} \quad \text{golden ratio}$$

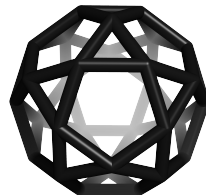
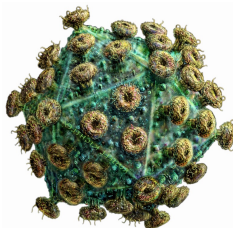
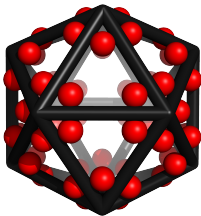
$$\tau = \frac{1}{2}(1 + \sqrt{5}) = 2 \cos \frac{\pi}{5}$$

$$x^2 = x + 1$$

$$\tau' = \sigma = \frac{1}{2}(1 - \sqrt{5}) = 2 \cos \frac{2\pi}{5}$$

$$\tau + \sigma = 1, \tau\sigma = -1$$

The Icosahedron



- **Rotational** icosahedral group is $I = A_5$ of order **60**
- **Full** icosahedral group is H_3 of order **120** (including reflections/inversion); generated by the root system icosidodecahedron

Clifford Algebra and orthogonal transformations

- Form an algebra using the **Geometric Product** for two vectors

$$ab \equiv a \cdot b + a \wedge b$$

- Inner product** is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting a in b is given by $a' = a - 2(a \cdot b)b = -bab$ (b and $-b$ **doubly cover** the same reflection)
- Via **Cartan-Dieudonné** theorem any orthogonal (/conformal/modular) transformation can be written as **successive reflections**

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1 = \pm A x \tilde{A}$$

Clifford Algebra of 3D

- E.g. **Pauli algebra** in 3D (likewise for **Dirac algebra** in 4D) is

$$\begin{array}{cccc}
 \underbrace{\{1\}} & \underbrace{\{e_1, e_2, e_3\}} & \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}} & \underbrace{\{I \equiv e_1 e_2 e_3\}} \\
 1 \text{ scalar} & 3 \text{ vectors} & 3 \text{ bivectors} & 1 \text{ trivector}
 \end{array}$$

- We can **multiply together root vectors** in this algebra $\alpha_i \alpha_j \dots$
- A general element has **8** components, **even** products (rotations/spinors) have **four** components:

$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow R \tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

- So behaves as a **4D Euclidean** object – inner product

$$(R_1, R_2) = \frac{1}{2}(R_2 \tilde{R}_1 + R_1 \tilde{R}_2)$$

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Induction Theorem – root systems

- Theorem: 3D spinor groups give 4D root systems.

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2. $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

Induction Theorem – root systems

- Theorem: **3D spinor groups** give **4D root systems**.
- Check axioms:
 1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$
 2. $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$
- Proof: 1. **R and $-R$** are in a spinor group by construction (**double cover** of orthogonal transformations), 2. closure under reflections is guaranteed by the **closure property of the spinor group** (with a twist: $-R_1 \tilde{R}_2 R_1$)
- Induction Theorem: **Every rank-3 root system induces a rank-4 root system** (and thereby **Coxeter groups**)

Spinors from reflections

- The 6 **roots** in $A_1 \times A_1 \times A_1$ generate 8 **spinors**.
- $\pm e_1, \pm e_2, \pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The **discrete spinor group** is isomorphic to the **quaternion** group Q .
- As 4D vectors these are the 8 **roots** of $A_1 \times A_1 \times A_1 \times A_1$ (the 16-cell).

H_4 as a rotation group I: as icosahedral spinors

- The H_3 root system has 30 **roots** e.g. simple roots $\alpha_1 = e_2, \alpha_2 = -\frac{1}{2}((\tau - 1)e_1 + e_2 + \tau e_3)$ and $\alpha_3 = e_3$.
- The subgroup of **rotations** is A_5 of order **60**
- These are doubly covered by **120** spinors of the form $\alpha_1 \alpha_2 = -\frac{1}{2}(1 - (\tau - 1)e_1 e_2 + \tau e_2 e_3)$, $\alpha_1 \alpha_3 = e_2 e_3$ and $\alpha_2 \alpha_3 = -\frac{1}{2}(\tau - (\tau - 1)e_3 e_1 + e_2 e_3)$.
- As a set of **vectors** in 4D, they are

$(\pm 1, 0, 0, 0)$ (8 permutations) , $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ (16 permutations)

$\frac{1}{2}(0, \pm 1, \pm \sigma, \pm \tau)$ (96 even permutations) ,

which are precisely the 120 roots of the **H_4 root system**.

Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the **Platonic Solids**:
- The 6/12/18/30 **roots** in $A_1 \times A_1 \times A_1 / A_3 / B_3 / H_3$ generate 8/24/48/120 **spinors**.
- E.g. $\pm e_1, \pm e_2, \pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The **discrete spinor group** is isomorphic to the **quaternion group** Q / **binary tetrahedral group** $2T$ / **binary octahedral group** $2O$ / **binary icosahedral group** $2I$).

A_1^3	A_3	B_3	H_3
A_1^4	D_4	F_4	H_4

Exceptional Root Systems

- **Exceptional** phenomena: D_4 (**triality**, important in string theory), F_4 (**largest lattice symmetry** in 4D), H_4 (**largest non-crystallographic symmetry**); **Exceptional** D_4 and F_4 arise from **series** A_3 and B_3

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$	
A_3		$2T$	D_4	
B_3		$2O$	F_4	
H_3		$2I$	H_4	

Arnold's Trinities

Arnold's observation that many areas of real mathematics can be **complexified** and **quaternionified** resulting in theories with a similar structure.

- The **fundamental trinity** is thus $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The **projective spaces** $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The **spheres** $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The **Möbius/Hopf bundles** $(S^1 \rightarrow S^1, S^4 \rightarrow S^2, S^7 \rightarrow S^4)$
- The **Lie Algebras** (E_6, E_7, E_8)
- The symmetries of the **Platonic Solids** (A_3, B_3, H_3)
- The **4D groups** (D_4, F_4, H_4)
- **New connections** via my **Clifford spinor construction** (see McKay correspondence)

Platonic Trinities

- Arnold's connection between (A_3, B_3, H_3) and (D_4, F_4, H_4) is **very convoluted** and involves numerous other trinities at intermediate steps:
- **Decomposition of the projective plane** into Weyl chambers and Springer cones
- The **number of Weyl chambers** in each segment is $24 = 2(1 + 3 + 3 + 5)$, $48 = 2(1 + 5 + 7 + 11)$, $120 = 2(1 + 11 + 19 + 29)$
- Notice this miraculously **matches the quasihomogeneous weights** $((2, 4, 4, 6), (2, 6, 8, 12), (2, 12, 20, 30))$ of the Coxeter groups (D_4, F_4, H_4)
- Believe the Clifford connection is **more direct**

A unified framework for polyhedral groups

Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm \tilde{A}xA$
$Spin(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$Pin(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the **chiral icosahedral** group has 60 elements, encoded in Clifford by 120 spinors, which form the **binary icosahedral** group
- together with the **inversion/pseudoscalar** I this gives 60 rotations and 60 rotoinversions, i.e. the **full icosahedral** group H_3 in 120 elements (with 240 pinors)

Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate **conjugacy classes** etc of pinors in Clifford algebra
- Chiral (**binary**) polyhedral groups have irreps
- tetrahedral (12/24): 1, 1', 1'', 2_s , $2'_s$, $2''_s$, 3
- octahedral (24/48): 1, 1', 2, 2_s , $2'_s$, 3, 3', 4_s
- icosahedral (60/120): 1, 2_s , $2'_s$, 3, $\bar{3}$, 4, 4_s , 5, 6_s
- Binary groups are **discrete subgroups of $SU(2)$** and all thus have a 2_s spinor irrep
- Connection with the **McKay correspondence**!

The McKay Correspondence: Coxeter number, dimensions of irreps and tensor product graphs

binary polyhedral groups
 $2T, 2O, 2I$
 $\sum d_i$ 12, 18, 30
 $\sum d_i^2$ 24, 48, 120

McKay correspondence

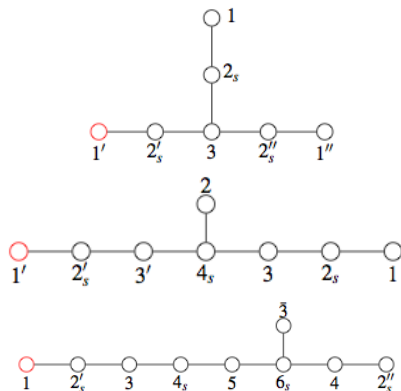
Exceptional
Lie Groups

E_6 , 12

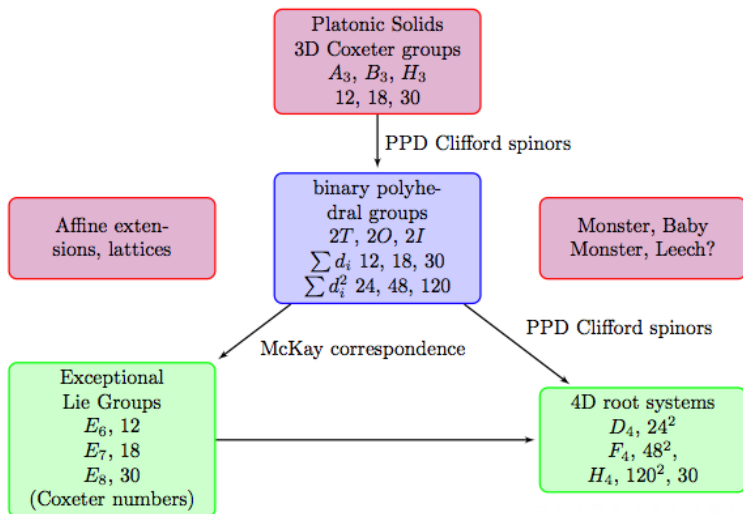
E_7 , 18

E_8 , 30

(Coxeter numbers)



The McKay Correspondence



The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the **cyclic** and **dicyclic groups** are in correspondence with A_n and D_n , e.g. the quaternion group Q and D_4^+ . So McKay correspondence not just a trinity but **ADE-classification**. We also have $I_2(n)$ on top of the trinity (A_3, B_3, H_3)

rank-3 group	diagram	binary	rank-4 group	diagram	Lie algebra	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$		D_4^+	
A_3		$2T$	D_4		E_6^+	
B_3		$2O$	F_4		E_7^+	
H_3		$2I$	H_4		E_8^+	

An indirect connection between E_8 and H_3 ?

- Trinities:

$(12, 18, 30)$

(A_3, B_3, H_3)

$(2T, 2O, 2I)$

(D_4, F_4, H_4)

(E_6, E_7, E_8)

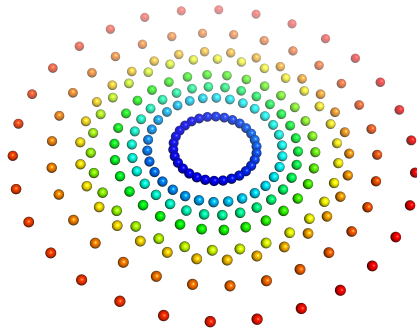
4D geometry is surprisingly important for HEP

- 4D root systems are **surprisingly relevant to HEP**
- A_4 is $SU(5)$ and comes up in **Grand Unification**
- D_4 is $SO(8)$ and is the little group of **String theory**
- In particular, its **triality symmetry** is crucial for showing the equivalence of RNS and GS strings
- B_4 is $SO(9)$ and is the little group of **M-Theory**
- F_4 is the **largest crystallographic** symmetry in 4D and H_4 is the **largest non-crystallographic** group
- The above are **subgroups** of the latter two
- **Spinorial nature** of the root systems could have **surprising consequences for HEP**

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Exceptional E_8 (projected into the Coxeter plane)

- E_8 root system has 240 roots, H_3 has order 120



Exceptional E_8 – from the icosahedron

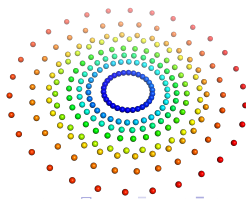
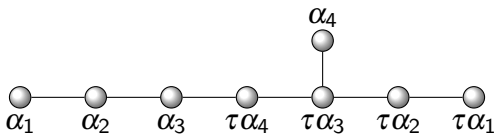
- Saw **even** products of the 30 roots of H_3 gave 120 **spinors** which in turn gave H_4 root system
- Taking **all** products gives group of 240 **pinors** with 8 components
- Essentially the **inversion** / just doubles the spinors

$$\begin{array}{cccc}
 \underbrace{\{1\}} & \underbrace{\{e_1, e_2, e_3\}} & \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}} & \underbrace{\{I \equiv e_1 e_2 e_3\}} \\
 1 \text{ scalar} & 3 \text{ vectors} & 3 \text{ bivectors} & 1 \text{ trivector}
 \end{array}$$

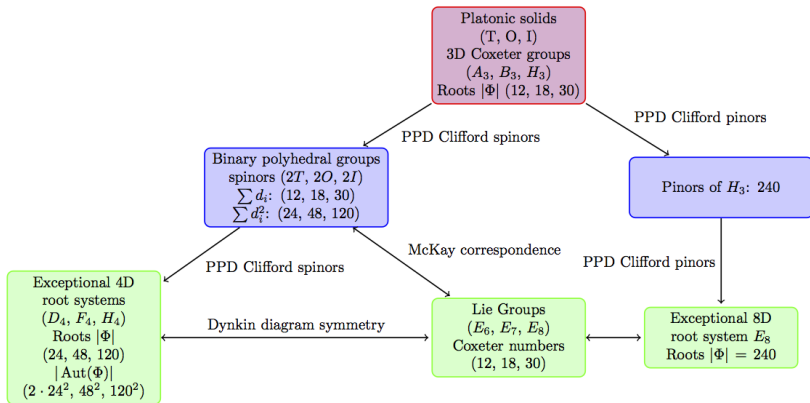
$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \text{ and } IR = b_0 e_1 e_2 e_3 + b_1 e_1 + b_2 e_2 + b_3 e_3$$

- Most intuitive inner product on the pinors gives only $H_4 \oplus H_4$
- But slightly more technical **inner product** gives precisely the **E_8 root system** from the **icosahedron**!

- Order 120 group H_3 doubly covered by 240 (s)pinors
- Essentially $H_4 + IH_4$, two sets of 120
- Multiply second set by τI , take inner products, taking into account $\tau^2 = \tau + 1$, but THEN: set $\tau \rightarrow 0$! Each inner product is $(\alpha_i, \alpha_j) = a + \tau b \rightarrow (\alpha_i, \alpha_j)_\tau := a$ (R. Wilson's reduced inner product)
- Like the other exceptional geometries, E_8 is actually hidden within 3D geometry!

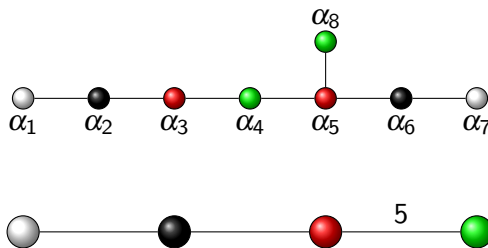


New, explicit connections



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Projection and Diagram Foldings

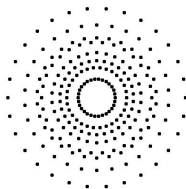
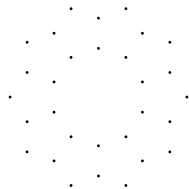
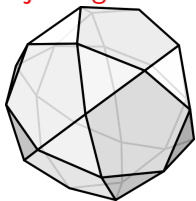


$$s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}, \quad s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}, \quad s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}, \quad s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4$$

- E_8 has a H_4 subgroup of **rotations** via a 'partial folding'
- Can **project** 240 E_8 roots to $H_4 + \tau H_4$ – essentially the **reverse** of the previous construction!
- Coxeter element & number** of E_8 and H_4 are the **same**

The Coxeter Plane

- Can show **every** (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by **projecting** their root system onto the Coxeter plane



Coxeter Elements, Degrees and Exponents

- Like the symmetric group, Coxeter groups can have **invariant polynomials**. Their **degrees** d are important invariants/group characteristics.
- Turns out that actually **degrees** d are intimately related to so-called **exponents** m $m = d - 1$.

Coxeter Elements, Degrees and Exponents

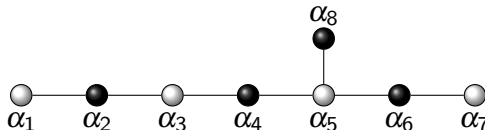
- A **Coxeter Element** is any combination of all the simple reflections $w = s_1 \dots s_n$, i.e. in Clifford algebra it is encoded by the versor $W = \alpha_1 \dots \alpha_n$ acting as $v \rightarrow wv = \pm \tilde{W} v W$. All such elements are conjugate and thus their **order** is invariant and called the **Coxeter number** h .
- The Coxeter element has **complex eigenvalues** of the form $\exp(2\pi mi/h)$ where m are called **exponents**:
 $w x = \exp(2\pi mi/h) x$
- Standard theory **complexifies** the real Coxeter group situation in order to find **complex eigenvalues**, then takes **real sections** again (the unfortunate standard procedure in many situations) – without any insight into the complex structure (or in fact, there are different ones).

Coxeter Elements, Degrees and Exponents

- The Coxeter element has **complex eigenvalues** of the form $\exp(2\pi mi/h)$ where m are called **exponents**
- Standard theory **complexifies** the real Coxeter group situation in order to find **complex eigenvalues**, then takes **real** sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure(s)
- In particular, **1** and **$h-1$** are always exponents
- Turns out that actually **exponents and degrees** are intimately related ($m = d-1$). The construction is slightly roundabout but uniform, and uses the **Coxeter plane**.

The Coxeter Plane

- In particular, can show **every** (for our purposes) Coxeter group has a Coxeter plane
- Existence relies on the fact that all groups in question have **tree-like Dynkin diagrams**, and thus admit an **alternate colouring**
- Essentially just gives **two sets of mutually commuting generators**



The Coxeter Plane

- Existence relies on the fact that all groups in question have **tree-like Dynkin diagrams**, and thus admit an alternate colouring
- Essentially just gives **two sets of orthogonal = mutually commuting generators but anticommuting root vectors** α_w and α_b (duals ω)
- Cartan matrices are positive definite, and thus have a **Perron-Frobenius** (all positive) eigenvector λ_i .
- Take **linear combinations** of components of this eigenvector as coefficients of two vectors from the orthogonal sets

$$v_w = \sum \lambda_w \omega_w \text{ and } v_b = \sum \lambda_b \omega_b$$
- Their **outer product/Coxeter plane bivector** $B_C = v_b \wedge v_w$ describes an **invariant plane** where w acts by rotation by $2\pi/h$.

Clifford Algebra and the Coxeter Plane – 2D case

$$I_2(n) \quad \circ \xrightarrow{n} \circ$$

- For $I_2(n)$ take $\alpha_1 = e_1, \alpha_2 = -\cos \frac{\pi}{n} e_1 + \sin \frac{\pi}{n} e_2$

- So **Coxeter versor** is just

$$W = \alpha_1 \alpha_2 = -\cos \frac{\pi}{n} + \sin \frac{\pi}{n} e_1 e_2 = -\exp \left(-\frac{\pi}{n} \right)$$

- In Clifford algebra it is therefore immediately obvious that the action of the $I_2(n)$ Coxeter element is described by a versor (here a rotor/spinor) that encodes **rotations in the $e_1 e_2$ -Coxeter-plane** and yields $h = n$ since trivially $W^n = (-1)^{n+1}$ yielding $w^n = 1$ via $wv = \tilde{W}vW$.

Clifford Algebra and the Coxeter Plane – 2D case

- So **Coxeter versor** is just $W = -\exp\left(-\frac{\pi I}{n}\right)$
- $I = e_1 e_2$ **anticommutes** with both e_1 and e_2 such that **sandwiching formula** becomes

$$v \rightarrow wv = \tilde{W}vW = \tilde{W}^2v = \exp\left(\pm \frac{2\pi I}{n}\right)v \text{ immediately}$$

yielding the standard result for the **complex eigenvalues** in real Clifford algebra **without any need for artificial complexification**

- The Coxeter plane bivector $B_C = e_1 e_2 = I$ gives the **complex structure**
- The Coxeter plane bivector B_C is invariant under the **Coxeter versor** $\tilde{W}B_CW = \pm B_C$.

Clifford Algebra and the Coxeter Plane – 3D case

- In 3D, A_3 , B_3 , H_3 have $\{1, 2, 3\}$, $\{1, 3, 5\}$ and $\{1, 5, 9\}$
- Coxeter element is product of a **spinor** in the Coxeter plane with the same complex structure as before, and a **reflection perpendicular** to the plane
- So in 3D still completely determined by the plane
- **1** and **$h - 1$** are **rotations** in **Coxeter plane**
- **$h/2$** is the **reflection** (for v in the normal direction)

$$wv = \tilde{W}^2 = \exp\left(\pm \frac{2\pi I}{h} \frac{h}{2}\right) = \exp(\pm \pi I)v = -v$$

Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can **factorise** W into **orthogonal** (commuting/anticommuting) components

$$W = \alpha_1 \dots \alpha_n = W_1 \dots W_n \text{ with } W_i = \exp(\pi m_i l_i / h)$$
- Here, l_i is a bivector describing a **plane** with $l_i^2 = -1$
- For v **orthogonal to the plane** described by l_i we have

$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i W_i v = v \text{ so cancels out}$$
- For v **in the plane** we have

$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i l_i / h) v$$
- Thus if we **decompose** W into **orthogonal eigenspaces**, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

Clifford algebra: no need for complexification

- For v in the plane we have

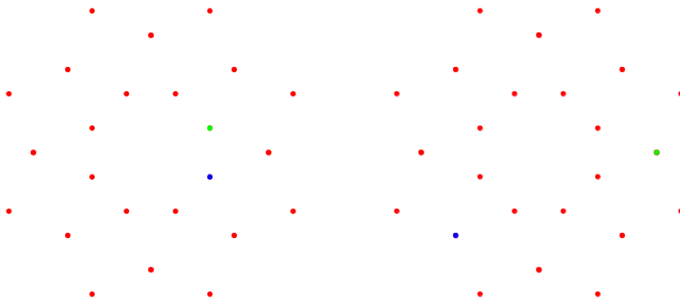
$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i / h) v$$

- So **complex eigenvalue equation** arises geometrically **without any need** for complexification
- Different complex structures** immediately give different **eigenplanes**
- Eigenvalues/angles/**exponents** given from just factorising $W = \alpha_1 \dots \alpha_n$
- E.g. B_4 has exponents 1, 3, 5, 7 and $W = \exp\left(\frac{\pi}{8} I_1\right) \exp\left(\frac{3\pi}{8} I_2\right)$
- Here we have been looking for orthogonal eigenspaces, so **innocuous** – different complex structures commute
- But not in general – **naive complexification** can be misleading

4D case: B_4

- E.g. B_4 has exponents 1, 3, 5, 7
- Coxeter versor decomposes into **orthogonal components**

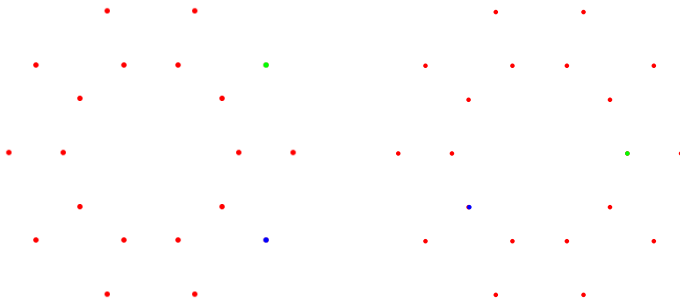
$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} I B_C\right)$$



4D case: A_4

- E.g. A_4 has exponents 1, 2, 3, 4
- Coxeter versor decomposes into **orthogonal components**

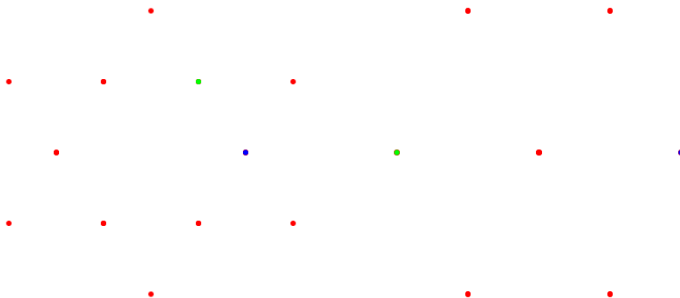
$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{5} B_C\right) \exp\left(\frac{2\pi}{5} I B_C\right)$$



4D case: D_4

- E.g. D_4 has exponents 1, 3, 3, 5
- Coxeter versor decomposes into **orthogonal components**

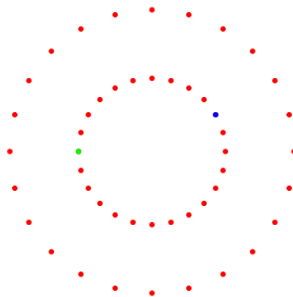
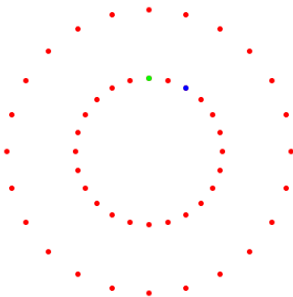
$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{3\pi}{6} I B_C\right)$$



4D case: F_4

- E.g. F_4 has exponents 1, 5, 7, 11
- Coxeter versor decomposes into **orthogonal components**

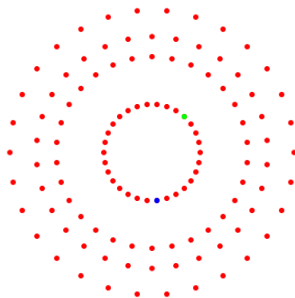
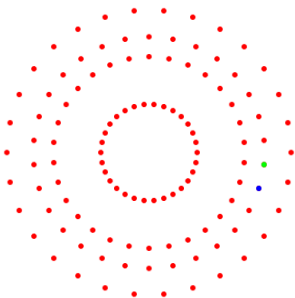
$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} I B_C\right)$$



4D case: H_4

- E.g. H_4 has exponents 1, 11, 19, 29
- Coxeter versor decomposes into **orthogonal components**

$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$$



Clifford Algebra and the Coxeter Plane – 4D case summary

rank 4	exponents	W-factorisation
A_4	1, 2, 3, 4	$W = \exp\left(\frac{\pi}{5} B_C\right) \exp\left(\frac{2\pi}{5} I B_C\right)$
B_4	1, 3, 5, 7	$W = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} I B_C\right)$
D_4	1, 3, 3, 5	$W = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{\pi}{2} I B_C\right)$
F_4	1, 5, 7, 11	$W = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} I B_C\right)$
H_4	1, 11, 19, 29	$W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$

Actually, in 2, 3 and 4 dimensions it couldn't really be any other way

Clifford Algebra and the Coxeter Plane – D_6

- For D_6 one has exponents $\boxed{1, 3, 5, 5, 7, 9}$
- Coxeter versor decomposes into orthogonal bits as

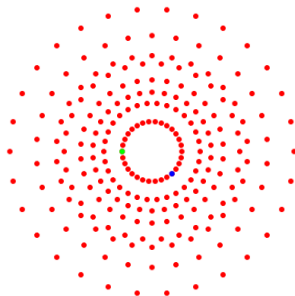
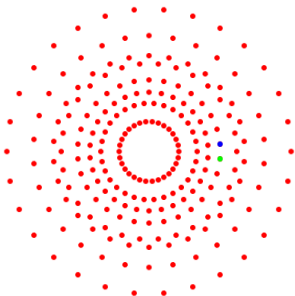
$$W = \frac{1}{\sqrt{5}}(e_1 + e_2 + e_3 - e_4 - e_5)e_6 \exp\left(\frac{\pi}{10}B_C\right) \exp\left(\frac{3\pi}{10}B_2\right)$$

- Now **bivector exponentials** correspond to **rotations in orthogonal planes**
- **Vector** factors correspond to **reflections**
- For odd n , there is always **one such vector factor** in D_n , and for even n there are **two**

8D case: E_8

- E.g. H_4 has exponents 1, 11, 19, 29, E_8 has 1, 7, 11, 13, 17, 19, 23, 29
- Coxeter versor decomposes into **orthogonal components**

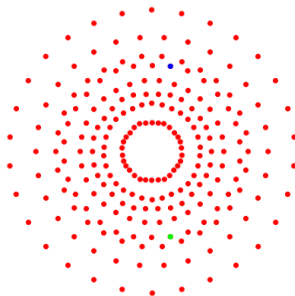
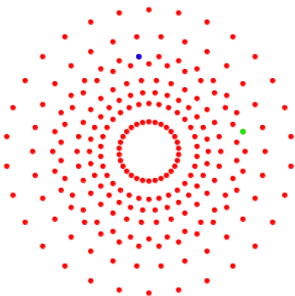
$$W = \alpha_1 \dots \alpha_8 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{7\pi}{30} B_2\right) \exp\left(\frac{11\pi}{30} B_3\right) \exp\left(\frac{13\pi}{30} B_4\right)$$



8D case: E_8

- E.g. H_4 has exponents 1, 11, 19, 29, E_8 has 1, 7, 11, 13, 17, 19, 23, 29
- Coxeter versor decomposes into **orthogonal components**

$$W = \alpha_1 \dots \alpha_8 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{7\pi}{30} B_2\right) \exp\left(\frac{11\pi}{30} B_3\right) \exp\left(\frac{13\pi}{30} B_4\right)$$



Imaginary differences – different imaginaries

So what has been **gained** by this **Clifford view**?

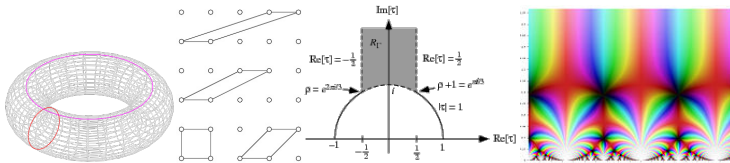
- There are **different** entities that serve as **unit imaginaries**
- They have a **geometric** interpretation as an **eigenplane of the Coxeter element**
- These don't need to **commute** with everything like i (though they do here – at least anticommute. But that is because we looked for **orthogonal decompositions**)
- But see that in general **naive complexification** can be a dangerous thing to do – **unnecessary**, issues of **commutativity**, **confusing** different imaginaries etc

Conclusions

- **All exceptional** geometries arise in **3D**, root systems giving rise to Lie groups/algebras etc
- Completely novel **spinorial** way of viewing the geometries as 3D phenomena – implications for HEP etc?
- More **natural** point of view, explaining **existence** and perhaps **automorphism groups**
- Unclear how one would see this in a **matrix framework** – might **require** Clifford point of view
- New view of Coxeter **degrees and exponents** with **geometric interpretation of imaginaries**
- A unified framework for doing **group and representation theory**: polyhedral, orthogonal, conformal, modular (Moonshine) etc

Thank you!

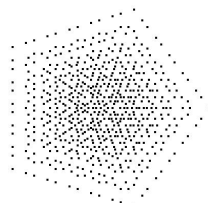
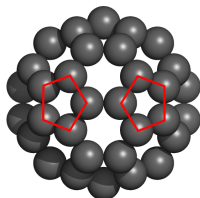
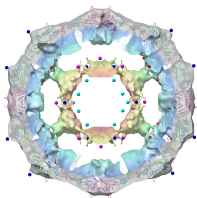
Modular group



- **Modular group**: interested in modular forms for applications in **Moonshine/string theory**: Monster 196883, Klein j 196884
- Modular generators: $T : \tau \rightarrow \tau + 1$, $S : \tau \rightarrow -1/\tau$
- $\langle S, T | S^2 = I, (ST)^3 = I \rangle$
- CGA: $T_X = 1 + \frac{ne_1}{2}$ and $S_X = e_1 e$
- $(S_X T_X)^3 = -1$ and $S_X^2 = 1$

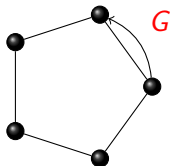
Motivation: Viruses

- Geometry of **polyhedra** described by **Coxeter** groups
- Viruses have to be 'economical' with their **genes**
- Encode **structure** modulo **symmetry**
- **Largest discrete symmetry of space** is the **icosahedral** group
- Many other 'maximally symmetric' objects in nature are also icosahedral: **Fullerenes & Quasicrystals**
- But: viruses are not just polyhedral – they have **radial structure**. **Affine extensions** give **translations**



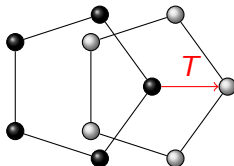
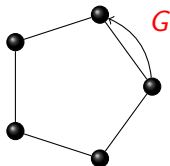
Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon



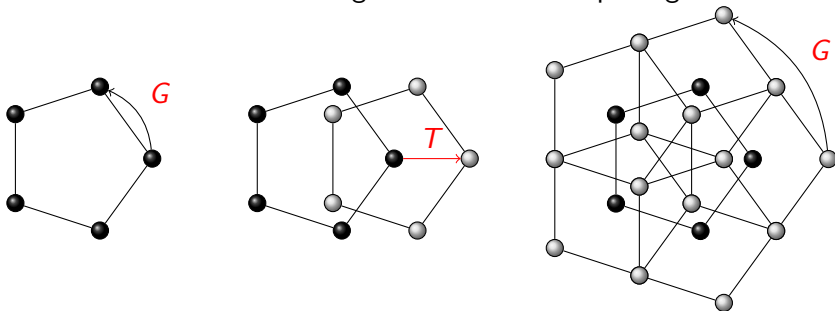
Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon



Affine extensions of non-crystallographic root systems

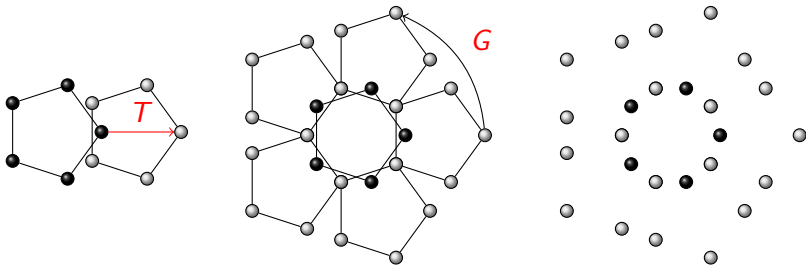
Unit translation along a vertex of a unit pentagon



A **random** translation would give 5 secondary pentagons, i.e. 25 points. Here we have **degeneracies** due to 'coinciding points'.

Affine extensions of non-crystallographic root systems

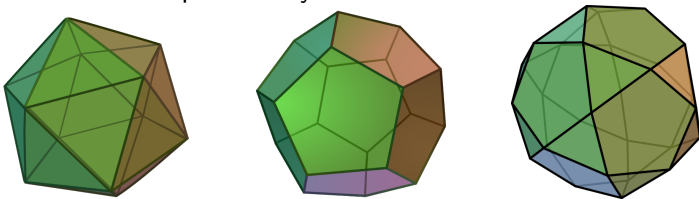
Translation of length $\tau = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ (golden ratio)



Looks like a **virus** or **carbon onion**

Extend icosahedral group with distinguished translations

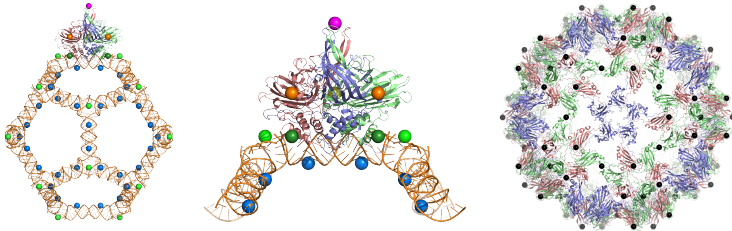
- Radial layers are **simultaneously constrained** by affine symmetry
- Works very well in practice: **finite library of blueprints**
- **Select** blueprint from the **outer shape** (capsid)
- Can **predict inner structure** (nucleic acid distribution) of the virus from the point array



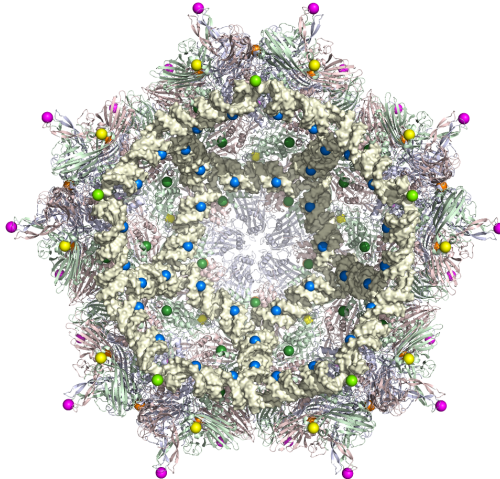
Affine extensions of the icosahedral group (giving translations) and their **classification**.

Use in Mathematical Virology

- Suffice to say **point arrays work very exceedingly well** in practice. Two papers on the mathematical (Coxeter) aspects.
- **Implemented computational problem in Clifford** – some **very interesting mathematics** comes out as well (see later).

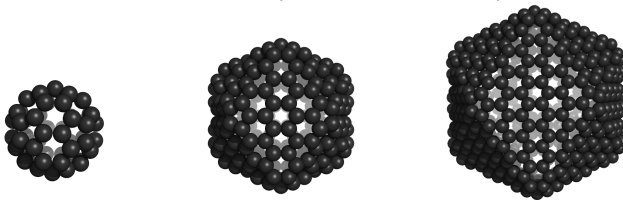


Use in Mathematical Virology



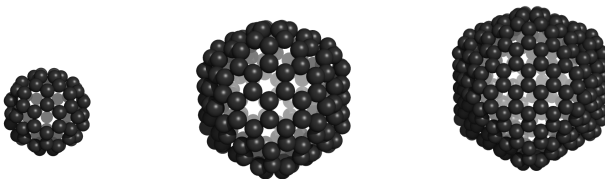
Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach: **carbon onions** ($C_{60} - C_{240} - C_{540}$)



Extension to fullerenes: carbon onions

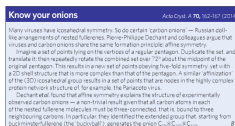
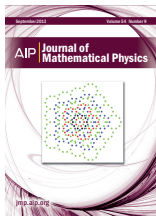
- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach: **carbon onions** ($C_{80} - C_{180} - C_{320}$)



References

- Novel Kac-Moody-type affine extensions of non-crystallographic Coxeter groups with Twarock/Bøhm J. Phys. A: Math. Theor. 45 285202 (2012)
- Affine extensions of non-crystallographic Coxeter groups induced by projection with Twarock/Bøhm Journal of Mathematical Physics 54 093508 (2013), [Cover article September](#)
- Viruses and Fullerenes – Symmetry as a Common Thread? with Twarock/Wardman/Keef March Cover Acta Crystallographica A 70 (2). pp. 162-167 (2014), and [Nature Physics Research Highlight](#)

Applications of affine extensions of non-crystallographic root systems



well-known effect for photons, and it turns out to hold for other quantum particles too. James Fickens and colleagues have performed the Hong-Ou-Mandel quantum interference experiment using plasmons, which are quantized surface plasma waves. Pairs of photons are fed into a specially designed photonic waveguide that mixes the paths of the light-excited surface plasmons to the same way as a beam splitter. The outcome is converted back into photons and measured by two detectors. As in the purely photonic case, the characteristic dip in coincidence rate is there, showing that the photons remain indistinguishable when they are converted into plasmons and interfere. 10

Written by May Chou, Kilo Gorgens, Albert Kopper, Bert Veldens and Alwin Wöhrle

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There are interesting applications to **quasicrystals**, **viruses** or **carbon onions**, but here concentrate on the **mathematical** aspects

Quaternions and Clifford Algebra

- The unit **spinors** $\{1; i; j; k\}$ of $Cl(3)$ are isomorphic to the **quaternion** algebra \mathbb{H} (up to sign)
- The 3D **Hodge dual of a vector** is a **pure bivector** which corresponds to a **pure quaternion**, and their products are identical (up to sign)

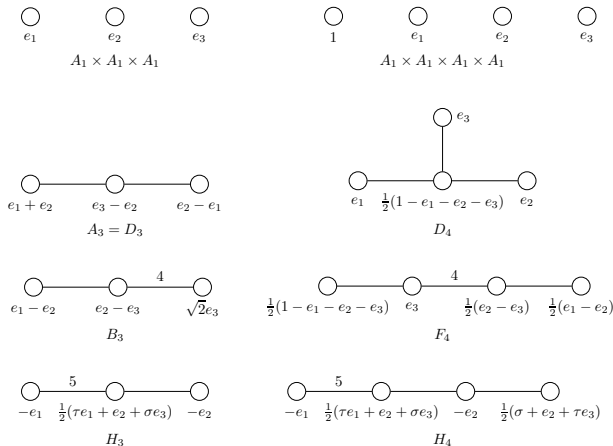
Discrete Quaternion groups

- The 8 quaternions of the form $(\pm 1, 0, 0, 0)$ and permutations are called the **Lipschitz units**, and form a realisation of the **quaternion group** in 8 elements.
- The 8 Lipschitz units together with $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ are called the **Hurwitz units**, and realise the **binary tetrahedral group** of order 24. Together with the 24 'dual' quaternions of the form $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$, they form a group isomorphic to the **binary octahedral group** of order 48.
- The 24 Hurwitz units together with the 96 unit quaternions of the form $(0, \pm \tau, \pm 1, \pm \sigma)$ and even permutations, are called the **Icosians**. The icosian group is isomorphic to the **binary icosahedral group** with 120 elements.

Quaternionic representations of 3D and 4D Coxeter groups

- Groups E_8 , D_4 , F_4 and H_4 have representations in terms of **quaternions**
- **Extensively used** in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why
- e.g. H_4 consists of 120 elements of the form $(\pm 1, 0, 0, 0)$, $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and $(0, \pm \tau, \pm 1, \pm \sigma)$
- Seen as remarkable that the **subset of the 30 pure quaternions** is a realisation of H_3 (**a sub-root system**)
- Similarly, A_3 , B_3 , $A_1 \times A_1 \times A_1$ have representations in terms of **pure quaternions**
- Will see there is a **much simpler geometric explanation**

Quaternionic representations used in the literature



Demystifying Quaternionic Representations

- 3D: **Pure quaternions** = Hodge dualised (pseudoscalar) **root vectors**
- In fact, they are the **simple roots of the Coxeter groups**
- 4D: **Quaternions** = disguised **spinors** – but those of the **3D Coxeter group** i.e. the binary polyhedral groups!
- This relation between 3D and 4D via the geometric product does not seem to be known
- Quaternion multiplication = ordinary Clifford reflections and rotations

Demystifying Quaternionic Representations

- Pure quaternion subset of 4D groups only gives 3D group if the 3D group contains the inversion/pseudoscalar /
- e.g. does not work for the tetrahedral group A_3 , but $A_3 \rightarrow D_4$ induction still works, with the central node essentially 'spinorial'
- In fact, it goes the other way around: the 3D groups induce the 4D groups via spinors
- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as $R_1 = \alpha_1 \alpha_2$ and $R_2 = \alpha_2 \alpha_3$
- Can see these are 'spinor generators' and how they don't really contain any more information/roots than the rank-3 groups alone

Quaternions vs Clifford versors

- **Sandwiching** is often seen as particularly nice feature of the quaternions giving rotations
- This is actually a **general feature** of Clifford algebras/versors in any dimension; the isomorphism to the quaternions is accidental to 3D
- However, the **root system** construction does not necessarily generalise
- 2D generalisation merely gives that $I_2(n)$ is **self-dual**
- **Octonionic** generalisation just induces two copies of the above 4D root systems, e.g. $A_3 \rightarrow D_4 \oplus D_4$