A 3D spinorial view of 4D exceptional phenomena

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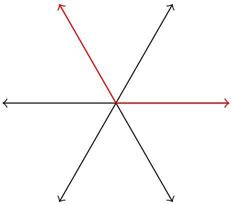
Overview

1. Introduction
   - Coxeter groups and root systems
   - Clifford algebras
   - ‘Platonic’ Solids

2. Combining Coxeter and Clifford
   - The Induction Theorem – from 3D to 4D
   - Automorphism Groups
   - Trinities and McKay correspondence

Root systems – A_2

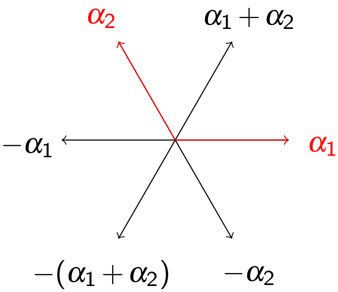


Root system Φ: set of vectors α such that

1. $\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \;\forall\, \alpha \in \Phi}$

2. $\boxed{s_\alpha \Phi = \Phi \;\forall\, \alpha \in \Phi}$

Root systems – A_2



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Simple roots: express every element of Φ via a $\mathbb{Z}$-linear combination (with coefficients of the same sign).

Coxeter groups

A Coxeter group is a group generated by some involutive generators $s_i, s_j \in S$ (i.e. $s_i^2 = 1$) subject to (mixed) relations of the form $\boxed{(s_i s_j)^{m_{ij}} = 1}$ with $\mathbb{Z} \ni m_{ij} = m_{ji} \geq 2$ for $i \neq j$.

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The finite Coxeter groups have a geometric representation where the involutions are realised as reflections at hyperplanes through the origin in a Euclidean vector space $\mathscr{E}$. In particular, let $(\cdot|\cdot)$ denote the inner product in $\mathscr{E}$, and v, $\alpha \in \mathscr{E}$.

The generator s_α corresponds to the reflection

$$\boxed{s_\alpha : v \to s_\alpha(v) = v - 2\frac{(v|\alpha)}{(\alpha|\alpha)}\alpha}$$

at a hyperplane perpendicular to the root vector α.

The action of the Coxeter group is to permute these root vectors.

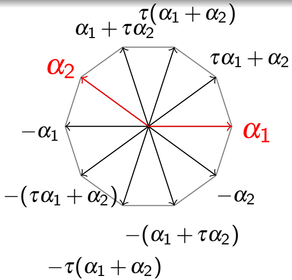
Cartan Matrices

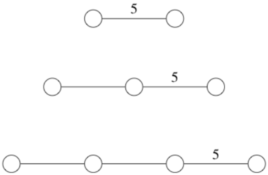
Cartan matrix of α_is is $\boxed{A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2\frac{|\alpha_j|}{|\alpha_i|}\cos\theta_{ij}}$

$$A_2\colon\ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal, simple link = roots at $\frac{\pi}{3}$, link with label m = angle $\frac{\pi}{m}$.

Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$





$$A=\begin{pmatrix}2 & -\tau\\ -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -\tau\\ 0 & -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -\tau\\ 0 & 0 & -\tau & 2\end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate rotational symmetries of order 5

linear combinations now in the extended integer ring

$\boxed{\mathbb{Z}[\tau]=\{a+\tau b | a,b\in\mathbb{Z}\}}$ golden ratio $\boxed{\tau=\frac{1}{2}(1+\sqrt{5})=2\cos\frac{\pi}{5}}$

$\boxed{x^2=x+1}$ $\boxed{\tau'=\sigma=\frac{1}{2}(1-\sqrt{5})=2\cos\frac{2\pi}{5}}$ $\boxed{\tau+\sigma=1, \tau\sigma=-1}$

Basics of Clifford Algebra I

- Form an algebra using the Geometric Product for two vectors

$$\boxed{ab \equiv a \cdot b + a \wedge b}$$

Basics of Clifford Algebra I

- Form an algebra using the Geometric Product for two vectors

$$\boxed{ab \equiv a \cdot b + a \wedge b}$$

- Extend via linearity and associativity to higher grade elements (multivectors)
- For an n-dimensional space generated by n orthogonal unit vectors e_i have 2^n elements
- Then $\boxed{e_i e_j = e_i \wedge e_j = -e_j e_i}$ so anticommute (Grassmann variables, exterior algebra)
- Unlike the inner and outer products separately, this product is invertible

Basics of Clifford Algebra II

- These are known to have matrix representations over the normed division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ $\Rightarrow$ Classification of Clifford algebras
- E.g. Pauli algebra in 3D (likewise for Dirac algebra in 4D) is

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

- These have the well-known matrix representations in terms of σ- and γ-matrices
- Working with these is not necessarily the most insightful thing to do, so here stress approach to work directly with the algebra

Reflections

- Clifford algebra is very efficient at performing reflections
- Consider reflecting the vector $\boxed{a = a_\perp + a_\parallel}$ in a hypersurface with unit normal n:

$$a' = a_\perp - a_\parallel = a - 2a_\parallel = a - 2(a\cdot n)n$$

- c.f. fundamental Weyl reflection $s_i : v \to s_i(v) = v - 2\frac{(v|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i$
- But in Clifford algebra have $a\cdot n = \frac{1}{2}(na + an)$ so reassembles into (note doubly covered by n and $-n$) sandwiching

$$\boxed{a' = -nan}$$

- So both Coxeter and Clifford frameworks are ideally suited to describing reflections – combine the two

Rotations

- Generate a rotation in the plane $m \wedge n$ when compounding two reflections wrt n then m:

$$a'' = mnanm \equiv Ra\tilde{R}$$

where $R = mn$ is called a rotor and a tilde denotes reversal of the order of the constituent vectors ($R\tilde{R} = 1$)

- Multivectors transform covariantly e.g.

$$MN \to (RM\tilde{R})(RN\tilde{R}) = RM\tilde{R}RN\tilde{R} = R(MN)\tilde{R}$$

so transform double-sidedly

- Spinors form a group, which gives a representation of the Spin group $Spin(n)$ – they transform single-sidedly (obvious it's a double (universal) cover)

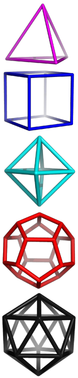
Geometric Algebra and orthogonal transformations

- Cartan-Dieudonné: every isometry is at most d reflections
- Since have a double cover of reflections (n and $-n$) we have a double cover of $O(p,q)$: $\mathrm{Pin}(p,q)$

$$\boxed{x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1}$$

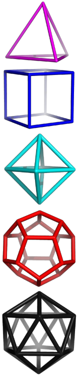
- Pinors = products of vectors $n_1 n_2 \dots n_k$ encode orthogonal transformations via 'sandwiching'
- Cartan-Dieudonné: rotations are an even number of reflections: $\mathrm{Spin}(p,q)$ doubly covers $SO(p,q)$

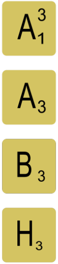
3D Platonic Solids



- There are 5 Platonic solids
- Tetrahedron (self-dual) (A_3)
- Dual pair octahedron and cube (B_3)
- Dual pair icosahedron and dodecahedron (H_3)
- Only the octahedron is a root system (actually for (A_1^3))

Clifford and Coxeter: Platonic Solids



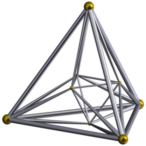
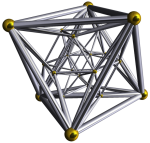
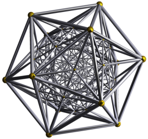
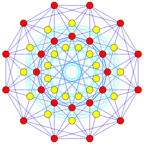


Platonic Solid	Group	root system
Tetrahedron	A_3 A_1^3	Cuboctahedron Octahedron
Octahedron Cube	B_3	Cuboctahedron +Octahedron
Icosahedron Dodecahedron	H_3	Icosidodecahedron

- Platonic Solids have been known for millennia
- Described by Coxeter groups

4D 'Platonic Solids'

- In 4D, there are 6 analogues of the Platonic Solids:
- 5-cell (self-dual) (A_4)
- Dual pair 16-cell and 8-cell (B_4)
- Dual pair 600-cell and 120-cell (H_4)
- 24-cell (self-dual) (D_4) – a 24-cell and its dual together are the F_4 root system
- These are 4D analogues of the Platonic Solids: regular convex 4-polytopes

4D 'Platonic Solids'

- 24-cell, 16-cell and 600-cell are all root systems, as is the related F_4 root system
- 8-cell and 120-cell are dual to a root system, so in 4D out of 6 Platonic Solids only the 5-cell (corresponding to A_n family) is not related to a root system!
- The 4D Platonic solids are not normally thought to be related to the 3D ones except for the boundary cells
- They have very unusual automorphism groups
- Some partial case-by-case algebraic results in terms of quaternions – here we show a uniform construction offering geometric understanding

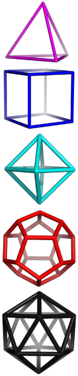
Mysterious Symmetries of 4D Polytopes

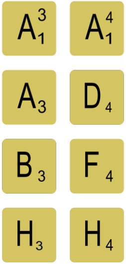
Spinorial symmetries

rank 4	$\|\Phi\|$	Symmetry
D_4 24-cell	24	$2 \cdot 24^2 = 576$
F_4 lattice	48	$48^2 = 2304$
H_4 600-cell	120	$120^2 = 14400$
A_1^4 16-cell	8	$3! \cdot 8^2 = 384$
$A_2 \oplus A_2$ prism	12	$12^2 = 144$
$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

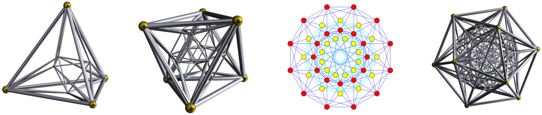
Similar for Grand Antiprism (H_4 without $H_2 \oplus H_2$) and Snub 24-cell ($2I$ without $2T$).

A new connection





- Platonic Solids have been known for millennia; described by Coxeter groups
- Concatenating reflections gives Clifford spinors (binary polyhedral groups)
- These induce 4D root systems
  $\psi = a_0 + a_i Ie_i \Rightarrow \psi\tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- 4D analogues of the Platonic Solids and give rise to 4D Coxeter groups



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2. Combining Coxeter and Clifford
   - The Induction Theorem – from 3D to 4D
   - Automorphism Groups
   - Trinities and McKay correspondence

Induction Theorem – root systems

- Theorem: 3D spinor groups give 4D root systems.
  1. $\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall\, \alpha \in \Phi}$
  2. $\boxed{s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi}$
- Proof: 1. R and $-R$ are in a spinor group by construction (double cover of orthogonal transformations), 2. closure under reflections is guaranteed by the closure property of the spinor group (with a twist: $-R_1\tilde{R}_2R_1$) via the norm $(R_1, R_2) = \frac{1}{2}(R_2\tilde{R}_1 + R_1\tilde{R}_2)$
- Induction Theorem: Every rank-3 root system induces a rank-4 root system (and thereby Coxeter groups)
- Counterexample: not every rank-4 root system is induced in this way

Induction Theorem – automorphism

- So induced 4D polytopes are actually root systems.
- Clear why the number of roots $|\Phi|$ is equal to $|G|$, the order of the spinor group
- Spinor group is trivially closed under conjugation, left and right multiplication. Results in non-trivial symmetries when viewed as a polytope/root system.
- Now explains symmetry of the polytopes/root system and thus the order of the rank-4 Coxeter group
- Theorem: The automorphism group of the induced root system contains two factors of the respective spinor group acting from the left and the right.

Recap: Clifford algebra and reflections & rotations

- Clifford algebra is very efficient at performing reflections via sandwiching

$$\boxed{a' = -nan}$$

- Generate a rotation when compounding two reflections wrt n then m (Cartan-Dieudonné theorem):

$$\boxed{a'' = mnanm \equiv Ra\tilde{R}}$$

where $R = mn$ is called a spinor and a tilde denotes reversal of the order of the constituent vectors ($R\tilde{R} = 1$)

Spinors from reflections

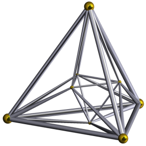
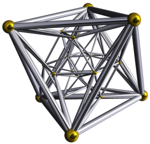
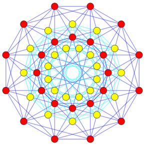
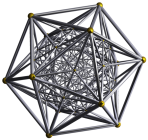
- The 3D Coxeter groups that are symmetry groups of the Platonic Solids:
- The 6 reflections in $A_1 \times A_1 \times A_1$ generate 8 spinors.
- $\pm e_1$, $\pm e_2$, $\pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The discrete spinor group is isomorphic to the quaternion group Q.

Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the Platonic Solids:
- The 6/12/18/30 reflections in $A_1 \times A_1 \times A_1/A_3/B_3/H_3$ generate 8/24/48/120 spinors.
- E.g. $\pm e_1$, $\pm e_2$, $\pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The discrete spinor group is isomorphic to the quaternion group Q / binary tetrahedral group $2T$/ binary octahedral group $2O$/ binary icosahedral group $2I$).

Spinors and Polytopes

- The space of $\mathrm{Cl}(3)$-spinors and quaternions have a 4D Euclidean signature: $\psi = a_0 + a_i I e_i \Rightarrow \psi\tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- Can reinterpret spinors in $\mathbb{R}^3$ as vectors in $\mathbb{R}^4$
- Then the spinors constitute the vertices of the 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell
- These are 4D analogues of the Platonic Solids: regular convex 4-polytopes

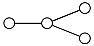





Exceptional Root Systems

- The 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell are in fact the root systems of $A_1 \times A_1 \times A_1 \times A_1$, D_4, F_4 and H_4
- Exceptional phenomena: D_4 (triality, important in string theory), F_4 (largest lattice symmetry in 4D), H_4 (largest non-crystallographic symmetry)
- Exceptional D_4 and F_4 arise from series A_3 and B_3
- In fact, as we have seen one can strengthen this statement on inducing polytopes to a statement on inducing root systems

Root systems in three and four dimensions

The spinors generated from the reflections contained in the respective rank-3 Coxeter group via the geometric product are realisations of the binary polyhedral groups Q, $2T$, $2O$ and $2I$, which were known to generate (mostly exceptional) rank-4 groups, but not known why, and why the 'mysterious symmetries'.

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$	$\circ \quad \circ \quad \circ$	Q	$A_1 \times A_1 \times A_1 \times A_1$	$\circ \quad \circ \quad \circ \quad \circ$
A_3	$\circ - \circ - \circ$	$2T$	D_4	
B_3	$\circ - \circ \overset{4}{-} \circ$	$2O$	F_4	$\circ - \circ \overset{4}{-} \circ - \circ$
H_3	$\circ - \circ \overset{5}{-} \circ$	$2I$	H_4	$\circ - \circ - \circ \overset{5}{-} \circ$

General Case of Induction

Only remaining case is what happens for $A_1 \oplus I_2(n)$ - this gives a doubling $I_2(n) \oplus I_2(n)$

rank 3	rank 4
A_3	D_4
B_3	F_4
H_3	H_4
A_1^3	A_1^4
$A_1 \oplus A_2$	$A_2 \oplus A_2$
$A_1 \oplus H_2$	$H_2 \oplus H_2$
$A_1 \oplus I_2(n)$	$I_2(n) \oplus I_2(n)$

Can do an analogous construction using 3 roots to generate a discrete octonion group. These are again root systems, however just two copies of the above.

Automorphism Groups

- So induced 4D polytopes are actually root systems via the binary polyhedral groups.
- Clear why the number of roots $|\Phi|$ is equal to $|G|$, the order of the spinor group.
- Spinor group is trivially closed under conjugation, left and right multiplication. Results in non-trivial symmetries when viewed as a polytope/root system.
- Now explains symmetry of the polytopes/root system and thus the order of the rank-4 Coxeter group
- Theorem: The automorphism group of the induced root system contains two factors of the respective spinor group acting from the left and the right.

Spinorial Symmetries of 4D Polytopes

Spinorial symmetries

rank 3	$\|\Phi\|$	$\|W\|$	rank 4	$\|\Phi\|$	Symmetry
A_3	12	24	D_4 24-cell	24	$2\cdot 24^2=576$
B_3	18	48	F_4 lattice	48	$48^2=2304$
H_3	30	120	H_4 600-cell	120	$120^2=14400$
A_1^3	6	8	A_1^4 16-cell	8	$3!\cdot 8^2=384$
$A_1\oplus A_2$	8	12	$A_2\oplus A_2$ prism	12	$12^2=144$
$A_1\oplus H_2$	12	20	$H_2\oplus H_2$ prism	20	$20^2=400$
$A_1\oplus I_2(n)$	$n+2$	$2n$	$I_2(n)\oplus I_2(n)$	$2n$	$(2n)^2$

Similar for Grand Antiprism (H_4 without $H_2\oplus H_2$) and Snub 24-cell ($2I$ without $2T$). Additional factors in the automorphism group come from 3D Dynkin diagram symmetries!

Some non-Platonic examples of spinorial symmetries

- Grand Antiprism: the 100 vertices achieved by subtracting 20 vertices of $H_2 \oplus H_2$ from the 120 vertices of the H_4 root system 600-cell – two separate orbits of $H_2 \oplus H_2$
- This is a semi-regular polytope with automorphism symmetry $\mathrm{Aut}(H_2 \oplus H_2)$ of order $400 = 20^2$
- Think of the $H_2 \oplus H_2$ as coming from the doubling procedure? (Likewise for $\mathrm{Aut}(A_2 \oplus A_2)$ subgroup)
- Snub 24-cell: $2T$ is a subgroup of $2I$ so subtracting the 24 corresponding vertices of the 24-cell from the 600-cell, one gets a semiregular polytope with 96 vertices and automorphism group $2T \times 2T$ of order $576 = 24^2$.

Arnold's Trinities

Arnold's observation that many areas of real mathematics can be complexified and quaternionified resulting in theories with a similar structure.

- The fundamental trinity is thus $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The projective spaces $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The spheres $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The Möbius/Hopf bundles $(S^1 \to S^1, S^4 \to S^2, S^7 \to S^4)$
- The Lie Algebras (E_6, E_7, E_8)
- The symmetries of the Platonic Solids (A_3, B_3, H_3)
- The 4D groups (D_4, F_4, H_4)
- New connections via my Clifford spinor construction (see McKay correspondence)

Platonic Trinities

- Arnold's connection between (A_3, B_3, H_3) and (D_4, F_4, H_4) is very convoluted and involves numerous other trinities at intermediate steps:
- Decomposition of the projective plane into Weyl chambers and Springer cones
- The number of Weyl chambers in each segment is $24 = 2(1+3+3+5), 48 = 2(1+5+7+11), 120 = 2(1+11+19+29)$
- Notice this miraculously matches the quasihomogeneous weights $((2,4,4,6),(2,6,8,12),(2,12,20,30))$ of the Coxeter groups (D_4, F_4, H_4)
- Believe the Clifford connection is more direct

A unified framework for polyhedral groups

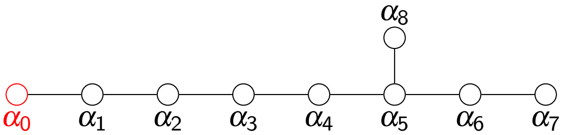
Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm\tilde{A}xA$
$\mathrm{Spin}(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$\mathrm{Pin}(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the chiral icosahedral group has 60 elements, encoded in Clifford by 120 spinors, which form the binary icosahedral group
- together with the inversion/pseudoscalar I this gives 60 rotations and 60 rotoinversions, i.e. the full icosahedral group H_3 in 120 elements (with 240 pinors)
- all three are interesting groups, e.g. in neutrino and flavour physics for family symmetry model building

Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate conjugacy classes etc of pinors in Clifford algebra
- Chiral (binary) polyhedral groups have irreps
- tetrahedral (12/24): $1, 1', 1'', 2_s, 2'_s, 2''_s, 3$
- octahedral (24/48): $1, 1', 2, 2_s, 2'_s, 3, 3', 4_s$
- icosahedral (60/120): $1, 2_s, 2'_s, 3, \bar{3}, 4, 4_s, 5, 6_s$
- Binary groups are discrete subgroups of $SU(2)$ and all thus have a 2_s spinor irrep
- Connection with the McKay correspondence!

Affine extensions – $E_8^{=}$



$$\boxed{-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8}$$

AKA E_8^{+} and along with E_8^{++} and E_8^{+++} thought to be the underlying symmetry of String and M-theory

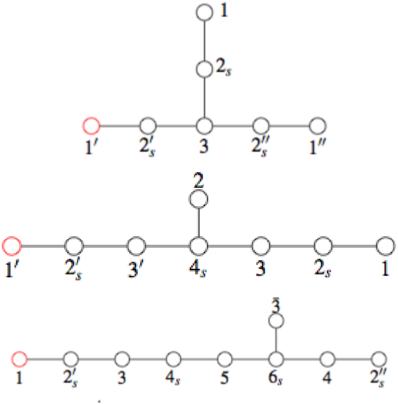
Also interesting from a pure mathematics point of view: E_8 lattice, McKay correspondence and Monstrous Moonshine.

The McKay Correspondence

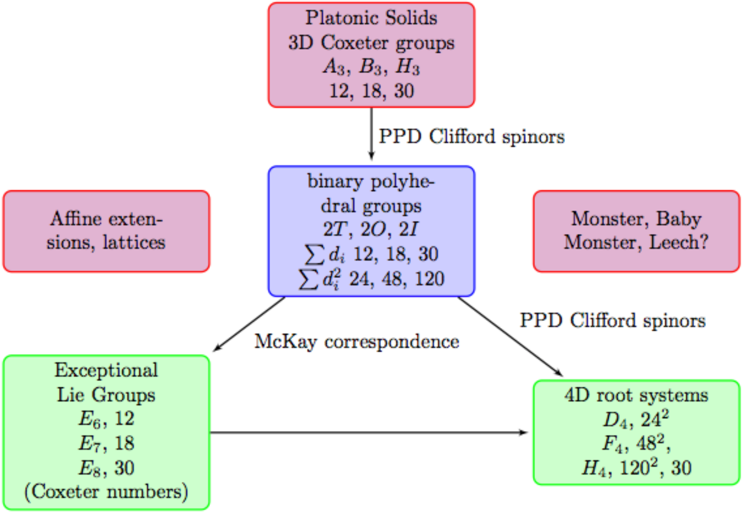
binary polyhedral groups
$2T$, $2O$, $2I$
$\sum d_i$ 12, 18, 30
$\sum d_i^2$ 24, 48, 120

McKay correspondence

Exceptional Lie Groups
E_6, 12
E_7, 18
E_8, 30
(Coxeter numbers)



The McKay Correspondence



The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the cyclic and dicyclic groups are in correspondence with A_n and D_n, e.g. the quaternion group Q and D_4^+. So McKay correspondence not just a trinity but ADE-classification. We also have $I_2(n)$ on top of the trinity (A_3, B_3, H_3)

rank-3 group	diagram	binary	rank-4 group	diagram	Lie algebra	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$		D_4^+	
A_3		$2T$	D_4		E_6^+	
B_3	4	$2O$	F_4	4	E_7^+	
H_3	5	$2I$	H_4	5	E_8^+	

4D geometry is surprisingly important for HEP

- 4D root systems are surprisingly relevant to HEP
- A_4 is $SU(5)$ and comes up in Grand Unification
- D_4 is $SO(8)$ and is the little group of String theory
- In particular, its triality symmetry is crucial for showing the equivalence of RNS and GS strings
- B_4 is $SO(9)$ and is the little group of M-Theory
- F_4 is the largest crystallographic symmetry in 4D and H_4 is the largest non-crystallographic group
- The above are subgroups of the latter two
- Spinorial nature of the root systems could have surprising consequences for HEP

Quaternions and Clifford Algebra

- The unit spinors $\{1; Ie_1; Ie_2; Ie_3\}$ of $\mathrm{Cl}(3)$ are isomorphic to the quaternion algebra $\mathbb{H}$
- The 3D Hodge dual of a vector is a pure bivector which corresponds to a pure quaternion, and their products are identical (up to sign)

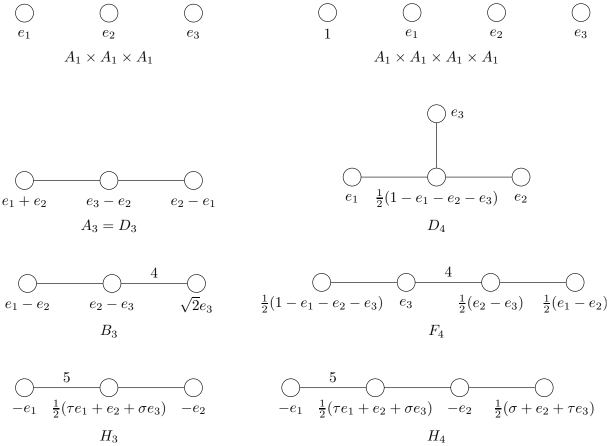
Discrete Quaternion groups

- The 8 quaternions of the form $(\pm 1, 0, 0, 0)$ and permutations are called the Lipschitz units, and form a realisation of the quaternion group in 8 elements.
- The 8 Lipschitz units together with $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ are called the Hurwitz units, and realise the binary tetrahedral group of order 24. Together with the 24 'dual' quaternions of the form $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$, they form a group isomorphic to the binary octahedral group of order 48.
- The 24 Hurwitz units together with the 96 unit quaternions of the form $(0, \pm\tau, \pm 1, \pm\sigma)$ and even permutations, are called the Icosians. The icosian group is isomorphic to the binary icosahedral group with 120 elements.

Quaternionic representations of 3D and 4D Coxeter groups

- Groups E_8, D_4, F_4 and H_4 have representations in terms of quaternions
- Extensively used in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why
- e.g. H_4 consists of 120 elements of the form $(\pm 1, 0, 0, 0)$, $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and $(0, \pm\tau, \pm 1, \pm\sigma)$
- Seen as remarkable that the subset of the 30 pure quaternions is a realisation of H_3 (a sub-root system)
- Similarly, B_3, $A_1 \times A_1 \times A_1$ have representations in terms of pure quaternions
- Will see there is a much simpler geometric explanation

Quaternionic representations used in the literature



Demystifying Quaternionic Representations

- 3D: Pure quaternions = Hodge dualised (pseudoscalar) root vectors
- In fact, they are the simple roots of the Coxeter groups
- 4D: Quaternions = disguised spinors – but those of the 3D Coxeter group i.e. the binary polyhedral groups!
- This relation between 3D and 4D via the geometric product does not seem to be known
- Quaternion multiplication = ordinary Clifford reflections and rotations

Demystifying Quaternionic Representations

- Pure quaternion subset of 4D groups only gives 3D group if the 3D group contains the inversion/pseudoscalar I
- e.g. does not work for the tetrahedral group A_3, but $A_3 \to D_4$ induction still works, with the central node essentially 'spinorial'
- In fact, it goes the other way around: the 3D groups induce the 4D groups via spinors
- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as $R_1 = \alpha_1\alpha_2$ and $R_2 = \alpha_2\alpha_3$
- Can see these are 'spinor generators' and how they don't really contain any more information/roots than the rank-3 groups alone

Quaternions vs Clifford versors

- Sandwiching is often seen as particularly nice feature of the quaternions giving rotations
- This is actually a general feature of Clifford algebras/versors in any dimension; the isomorphism to the quaternions is accidental to 3D
- However, the root system construction does not necessarily generalise
- 2D generalisation merely gives that $I_2(n)$ is self-dual
- Octonionic generalisation just induces two copies of the above 4D root systems, e.g. $A_3 \rightarrow D_4 \oplus D_4$

References (single-author)

- Clifford algebra unveils a surprising geometric significance of quaternionic root systems of Coxeter groups
  Advances in Applied Clifford Algebras, June 2013, Volume 23, Issue 2, pp 301-321
- A Clifford algebraic framework for Coxeter group theoretic computations (Conference Prize at AGACSE 2012)
  Advances in Applied Clifford Algebras 24 (1). pp. 89-108 (2014)
- Rank-3 root systems induce root systems of rank 4 via a new Clifford spinor construction arXiv:1207.7339 (2012)
- Platonic Solids generate their 4-dimensional analogues
  Acta Cryst. A69 (2013)

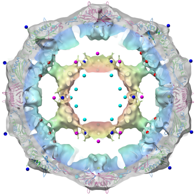
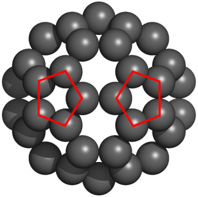
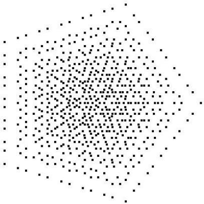
Conclusions

- Novel connection between geometry of 3D and 4D
- In fact, 3D seems more fundamental – contrary to the usual perspective of 3D subgroups of 4D groups
- Spinorial symmetries
- Clear why spinor group gives a root system and why two factors of the same group reappear in the automorphism group
- Novel spinorial perspective on 4D geometry
- Accidentalness of the spinor construction and exceptional 4D phenomena
- Connection with Arnold's trinities, the McKay correspondence and Monstrous Moonshine

Thank you!

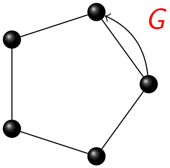
Motivation: Viruses

- Geometry of polyhedra described by Coxeter groups
- Viruses have to be 'economical' with their genes
- Encode structure modulo symmetry
- Largest discrete symmetry of space is the icosahedral group
- Many other 'maximally symmetric' objects in nature are also icosahedral: Fullerenes & Quasicrystals
- But: viruses are not just polyhedral – they have radial structure. Affine extensions give translations

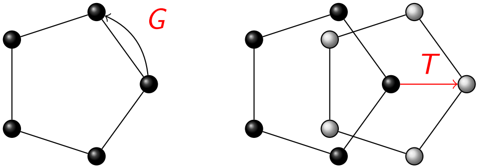
Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon



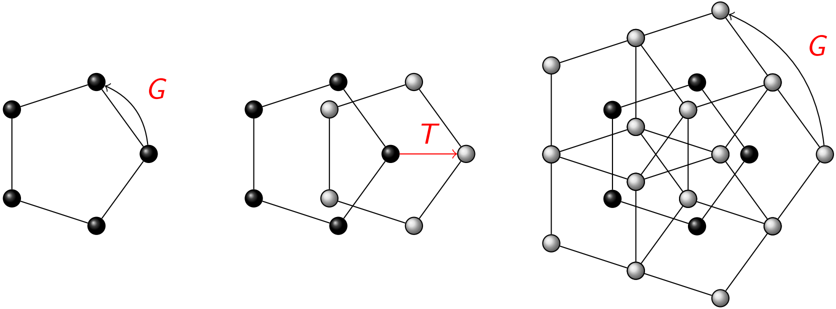
Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon



Affine extensions of non-crystallographic root systems

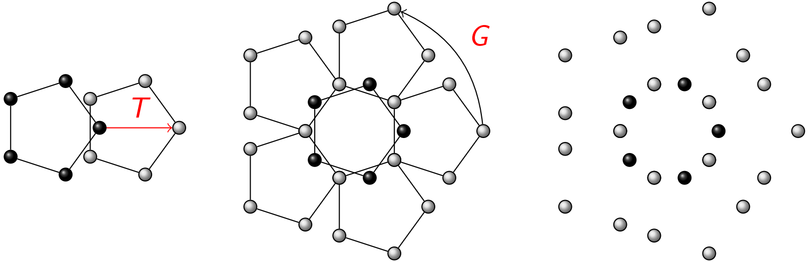
Unit translation along a vertex of a unit pentagon



A random translation would give 5 secondary pentagons, i.e. 25 points. Here we have degeneracies due to 'coinciding points'.

Affine extensions of non-crystallographic root systems

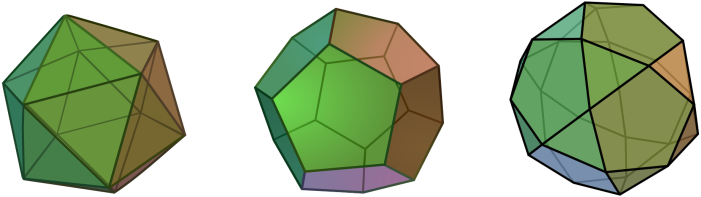
Translation of length $\tau = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$ (golden ratio)



Looks like a virus or carbon onion

Extend icosahedral group with distinguished translations

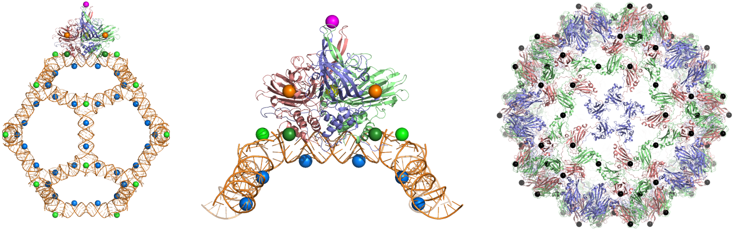
- Radial layers are simultaneously constrained by affine symmetry
- Works very well in practice: finite library of blueprints
- Select blueprint from the outer shape (capsid)
- Can predict inner structure (nucleic acid distribution) of the virus from the point array



Affine extensions of the icosahedral group (giving translations) and their classification.

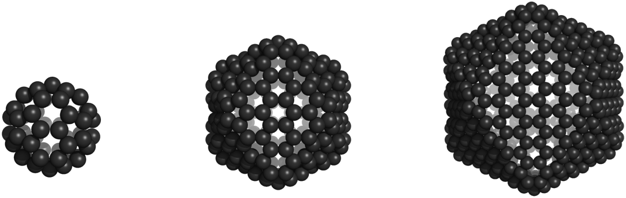
Use in Mathematical Virology

- Suffice to say point arrays work very exceedingly well in practice. Two papers on the mathematical (Coxeter) aspects.
- Implemented computational problem in Clifford – some very interesting mathematics comes out as well (see later).



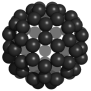
Extension to fullerenes: carbon onions

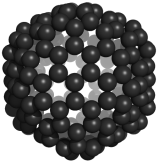
- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{60} - C_{240} - C_{540}$)

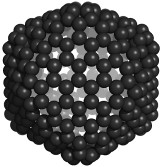


Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{80} - C_{180} - C_{320}$)







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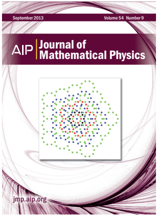
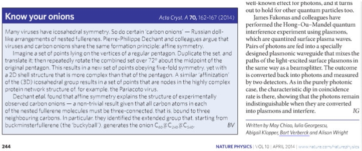
- Affine extensions of non-crystallographic Coxeter groups induced by projection with Twarock/Bœhm Journal of Mathematical Physics 54 093508 (2013), Cover article September
- Viruses and Fullerenes – Symmetry as a Common Thread? with Twarock/Wardman/Keef March Cover Acta Crystallographica A 70 (2). pp. 162-167 (2014), and Nature Physics Research Highlight

Applications of affine extensions of non-crystallographic root systems





Know your onions *Acta Cryst. A* **70,** 162-167 (2014)

Many viruses have icosahedral symmetry. So do certain 'carbon onions' — Russian doll-like arrangements of nested fullerenes. Pierre-Philippe Dechant and colleagues argue that viruses and carbon onions share the same formation principle: affine symmetry.

Imagine a set of points lying on the vertices of a regular pentagon. Duplicate the set, and translate it; then repeatedly rotate the combined set over 72° about the midpoint of the original pentagon. This results in a new set of points obeying five-fold symmetry, yet with a 2D shell structure that is more complex than that of the pentagon. A similar 'affinization' of the (3D) icosahedral group results in a set of points that are nodes in the highly complex protein network structure of, for example, the Pariacoto virus.

Dechant *et al.* found that affine symmetry explains the structure of experimentally observed carbon onions — a non-trivial result given that all carbon atoms in each of the nested fullerene molecules must be three-connected, that is, bound to three neighbouring carbons. In particular, they identified the extended group that, starting from buckminsterfullerene (the 'buckyball'), generates the onion $C_{60}@C_{240}@C_{540}$. *BV*

well-known effect for photons, and it turns out to hold for other quantum particles too.

James Fakonas and colleagues have performed the Hong–Ou–Mandel quantum interference experiment using plasmons, which are quantized surface plasma waves. Pairs of photons are fed into a specially designed plasmonic waveguide that mixes the paths of the light-excited surface plasmons in the same way as a beamsplitter. The outcome is converted back into photons and measured by two detectors. As in the purely photonic case, the characteristic dip in coincidence rate is there, showing that the photons remain indistinguishable when they are converted into plasmons and interfere. *IG*

Written by May Chiao, Iulia Georgescu, Abigail Klopper, Bart Verberck and Alison Wright

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There are interesting applications to quasicrystals, viruses or carbon onions, but here concentrate on the mathematical aspects