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The E_8 geometry from a Clifford perspective

Pierre-Philippe Dechant

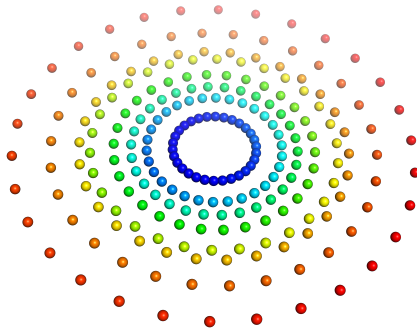
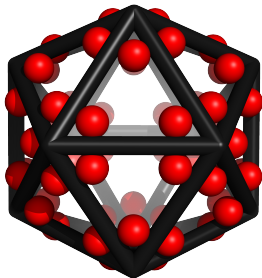
Mathematics Department, University of York

AGACSE Barcelona – July 29, 2015

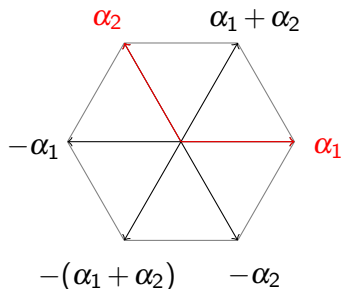
AGACSE 2012, La Rochelle: The birth of the E_8 question...



All exceptional geometries from 3D geometry



Root systems – A_2



reflection/Coxeter groups

$$s_\alpha : v \rightarrow s_\alpha(v) = v - 2 \frac{(v|\alpha)}{(\alpha|\alpha)} \alpha$$

Root system Φ : set of vectors α such that

1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$

2. $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

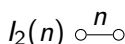
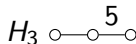
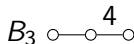
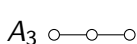
Simple roots: express every element of Φ via a \mathbb{Z} -linear combination (with coefficients of the same sign).

Cartan Matrices

Cartan matrix of α_i s is $A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2 \frac{|\alpha_j|}{|\alpha_i|} \cos \theta_{ij}$

$$A_2: A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal, simple link = roots at $\frac{\pi}{3}$, link with label $m =$ angle $\frac{\pi}{m}$.



Coxeter groups vs Lie groups vs Lie algebras vs root systems

- Lie group = **group** and **manifold** (e.g. spin groups: Doran, Hestenes et al)
- Lie algebra = **bilinear, antisymmetric bracket** and **Jacobi identity** (e.g. bivector algebras) = Lie group near the identity
- 'Nice' Lie algebras have **triangular decomposition**:

$$\mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+ : SU(2) : 1 + 1 + 1, E_8 : 120 + 8 + 120$$

- \mathcal{H} : **Cartan subalgebra/QN**; Creation & annihilation ops \mathcal{N} :
root lattice from crystallographic root systems
- **Weyl group** is a **crystallographic Coxeter group**:

$$A_n, B_n/C_n, D_n, G_2, F_4, E_6, E_7, E_8$$

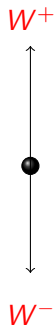
- So via this route always crystallographic! **Neglect**
 $I_2(n), H_3, H_4$.

Example – A_1 , $SU(2)$, Angular Momentum



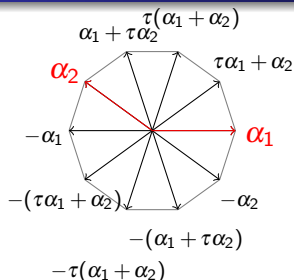
- Cartan subalgebra = Quantum number: L_z
- \mathcal{N}_+ : raising operator $L_+ = \alpha$
- \mathcal{N}_- : lowering operator $L_- = -\alpha$
- (L^2 is Casimir/commutes with all algebra elements, is however not actually in the algebra!)

Example – A_1 , $SU(2)$, Electroweak

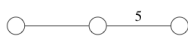


- Cartan subalgebra – Quantum number: A
- \mathcal{N}_+ : raising operator $W^+ = \alpha$
- \mathcal{N}_- : lowering operator $W^- = -\alpha$
- (Since SM electroweak is actually $SU(2) \times U(1)$, $U(1)$ gives another field i , such that physical Z^0 and γ are superpositions of A and i)
- Also W^\pm now charged and self-interact, unlike QED

Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$



$$A = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

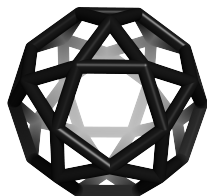
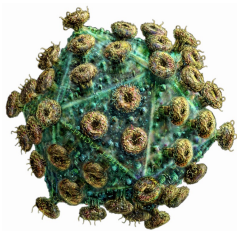
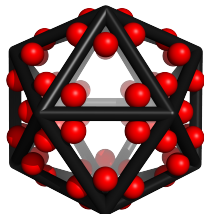
$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate **rotational symmetries of order 5**

linear combinations now in the **extended integer ring**

$$\mathbb{Z}[\tau] = \{a + \tau b \mid a, b \in \mathbb{Z}\} \quad \text{golden ratio} \quad \tau = \frac{1}{2}(1 + \sqrt{5}) = 2 \cos \frac{\pi}{5}$$

$$x^2 = x + 1 \quad \tau' = \sigma = \frac{1}{2}(1 - \sqrt{5}) = 2 \cos \frac{2\pi}{5} \quad \tau + \sigma = 1, \tau\sigma = -1$$

The Icosahedron



- **Rotational** icosahedral group is $I = A_5$ of order **60**
- **Full** icosahedral group is H_3 of order **120** (including reflections/inversion); generated by the root system icosidodecahedron

Clifford Algebra and orthogonal transformations

- Form an algebra using the **Geometric Product** for two vectors

$$ab \equiv a \cdot b + a \wedge b$$

- Inner product** is symmetric $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting a in b is given by $a' = a - 2(a \cdot b)b = -bab$ (b and $-b$ **doubly cover** the same reflection)
- Via **Cartan-Dieudonné** theorem any orthogonal/conformal/modular transformation can be written as **successive reflections**

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1 = \pm A x \tilde{A}$$

Clifford Algebra of 3D

- E.g. **Pauli algebra** in 3D (likewise for **Dirac algebra** in 4D) is

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1 e_2 e_3\}}_{1 \text{ trivector}}$$

- We can form the elements of the Coxeter groups by **multiplying together root vectors** in this algebra $\alpha_i \alpha_j \dots$
- General: 8 components, **even** products: (rotations/spinors)
four components:

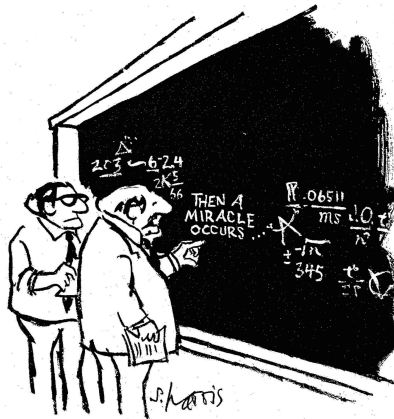
$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow R \tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

- So behaves as a **4D Euclidean** object – inner product

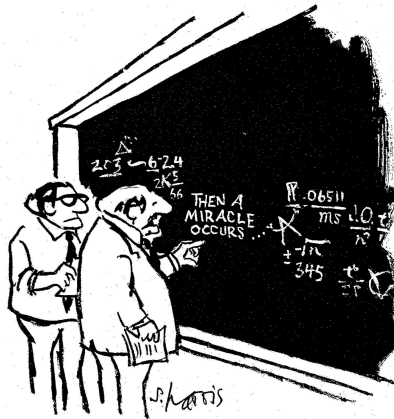
$$(R_1, R_2) = \frac{1}{2}(R_2 \tilde{R}_1 + R_1 \tilde{R}_2)$$

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4D from 3D

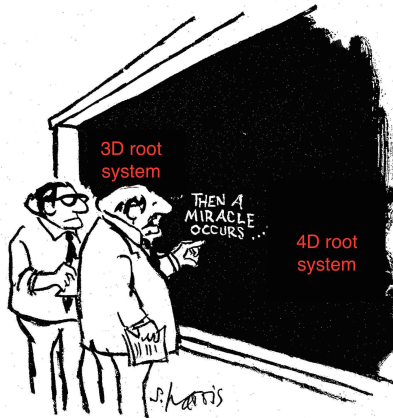


4D from 3D

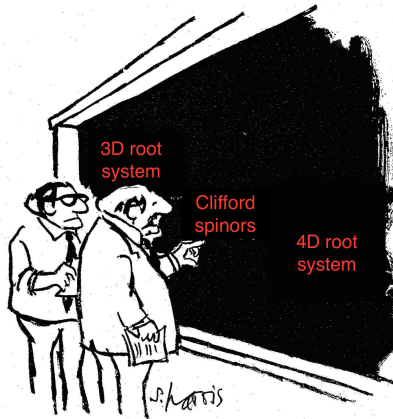


"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

4D from 3D



4D from 3D



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Induction Theorem – root systems

- Theorem: 3D spinor groups give 4D root systems.

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- Check axioms:

1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$

2. $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

Induction Theorem – root systems

- Theorem: **3D spinor groups** give **4D root systems**.
- Check axioms:
 1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$
 2. $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$
- Proof: 1. **R and $-R$** are in a spinor group by construction (**double cover** of orthogonal transformations), 2. closure under reflections is guaranteed by the **closure property of the spinor group** (with a twist: $-R_1 \tilde{R}_2 R_1$)
- Induction Theorem: **Every rank-3 root system induces a rank-4 root system** (and thereby **Coxeter groups**)
- Counterexample: **not every rank-4 root system** is induced in this way

Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the **Platonic Solids**:
- The 6 **reflections** in $A_1 \times A_1 \times A_1$ generate 8 **spinors**.
- $\pm e_1, \pm e_2, \pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The **discrete spinor group** is isomorphic to the **quaternion** group Q .

Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the **Platonic Solids**:
- The 6/12/18/30 **reflections** in $A_1 \times A_1 \times A_1/A_3/B_3/H_3$ generate 8/24/48/120 **spinors**.
- E.g. $\pm e_1, \pm e_2, \pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The **discrete spinor group** is isomorphic to the **quaternion group Q** / **binary tetrahedral group $2T$** / **binary octahedral group $2O$** / **binary icosahedral group $2I$**).

A_1^3	A_3	B_3	H_3
A_1^4	D_4	F_4	H_4

Exceptional Root Systems


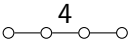
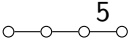
- The 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell are in fact the root systems of $A_1 \times A_1 \times A_1 \times A_1$, D_4 , F_4 and H_4
- Exceptional phenomena: D_4 (triality, important in string theory), F_4 (largest lattice symmetry in 4D), H_4 (largest non-crystallographic symmetry)
- Exceptional D_4 and F_4 arise from series A_3 and B_3
- In fact, as we have seen one can strengthen this statement on inducing polytopes to a statement on inducing root systems

3D vs 4D

- Have A_n , B_n and D_n families of root systems in **any dimension**
- In **3D**, have H_3 as an **accident** (icosahedron and dodecahedron)
- In **4D**, have F_4 and H_4 (and in some sense D_4) as accidents
- These 4D accidents have **unusual automorphism groups**
- Can **induce** all of these from the 3D cases, show they are **root systems** and explain their **automorphism groups**

Root systems in three and four dimensions

The **spinors** generated from the reflections contained in the respective **rank-3 Coxeter group** via the geometric product are realisations of the **binary polyhedral groups** Q , $2T$, $2O$ and $2I$, which were known to generate (mostly exceptional) **rank-4 groups**, but **not known why**, and why the '**mysterious symmetries**'.

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$	○ ○ ○	Q	$A_1 \times A_1 \times A_1 \times A_1$	○ ○ ○ ○
A_3	○—○—○	$2T$	D_4	
B_3	○—○—○ ⁴	$2O$	F_4	
H_3	○—○—○ ⁵	$2I$	H_4	

Induction Theorem – automorphism

- So induced **4D polytopes** are actually **root systems**.
- Clear why the **number of roots** $|\Phi|$ is equal to $|G|$, the **order of the spinor group**
- Spinor group is trivially **closed** under **conjugation, left and right multiplication**. Results in **non-trivial symmetries** when viewed as a **polytope/root system**.
- Now explains **symmetry** of the polytopes/root system and thus the **order** of the rank-4 Coxeter group
- Theorem: The **automorphism group** of the induced root system contains **two factors** of the respective spinor group acting from the **left** and the **right**.

Spinorial Symmetries of 4D Polytopes

Spinorial symmetries

rank 3	$ \Phi $	$ W $	rank 4	$ \Phi $	Symmetry
A_3	12	24	D_4 24-cell	24	$2 \cdot 24^2 = 576$
B_3	18	48	F_4 lattice	48	$48^2 = 2304$
H_3	30	120	H_4 600-cell	120	$120^2 = 14400$
A_1^3	6	8	A_1^4 16-cell	8	$3! \cdot 8^2 = 384$
$A_1 \oplus A_2$	8	12	$A_2 \oplus A_2$ prism	12	$12^2 = 144$
$A_1 \oplus H_2$	12	20	$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$A_1 \oplus I_2(n)$	$n+2$	$2n$	$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

Similar for **Grand Antiprism** (H_4 without $H_2 \oplus H_2$) and **Snub 24-cell** ($2I$ without $2T$). Additional factors in the automorphism group come from **3D Dynkin diagram symmetries!**

Arnold's Trinities

Arnold's observation that many areas of real mathematics can be **complexified** and **quaternionified** resulting in theories with a similar structure.

- The **fundamental trinity** is thus $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The **projective spaces** $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The **spheres** $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The **Möbius/Hopf bundles** $(S^1 \rightarrow S^1, S^4 \rightarrow S^2, S^7 \rightarrow S^4)$
- The **Lie Algebras** (E_6, E_7, E_8)
- The symmetries of the **Platonic Solids** (A_3, B_3, H_3)
- The **4D groups** (D_4, F_4, H_4)
- **New connections** via my **Clifford spinor construction** (see McKay correspondence)

Platonic Trinities

- Arnold's connection between (A_3, B_3, H_3) and (D_4, F_4, H_4) is **very convoluted** and involves numerous other trinities at intermediate steps:
- **Decomposition of the projective plane** into Weyl chambers and Springer cones
- The **number of Weyl chambers** in each segment is $24 = 2(1 + 3 + 3 + 5)$, $48 = 2(1 + 5 + 7 + 11)$, $120 = 2(1 + 11 + 19 + 29)$
- Notice this miraculously **matches the quasihomogeneous weights** $((2, 4, 4, 6), (2, 6, 8, 12), (2, 12, 20, 30))$ of the Coxeter groups (D_4, F_4, H_4)
- Believe the Clifford connection is **more direct**

A unified framework for polyhedral groups

Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm \tilde{A}xA$
$Spin(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$Pin(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the **chiral icosahedral** group has 60 elements, encoded in Clifford by 120 spinors, which form the **binary icosahedral** group
- together with the **inversion/pseudoscalar** I this gives 60 rotations and 60 rotoinversions, i.e. the **full icosahedral** group H_3 in 120 elements (with 240 pinors)
- all three are interesting groups, e.g. in **neutrino and flavour physics** for family symmetry model building

Some Group Theory: chiral, full, binary, pin

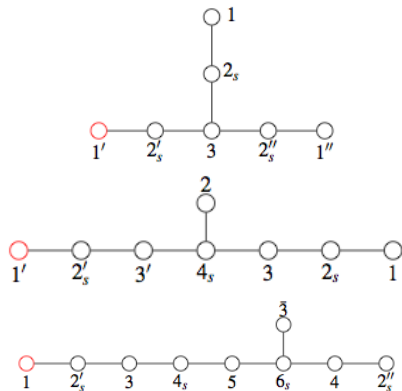
- Easy enough to calculate **conjugacy classes** etc of pinors in Clifford algebra
- Chiral (**binary**) polyhedral groups have irreps
- tetrahedral (12/24): 1, 1', 1'', 2_s , $2'_s$, $2''_s$, 3
- octahedral (24/48): 1, 1', 2, 2_s , $2'_s$, 3, 3', 4_s
- icosahedral (60/120): 1, 2_s , $2'_s$, 3, $\bar{3}$, 4, 4_s , 5, 6_s
- Binary groups are **discrete subgroups of $SU(2)$** and all thus have a 2_s spinor irrep
- Connection with the **McKay correspondence!**

The McKay Correspondence: Coxeter number, dimensions of irreps and (tensor product) graphs

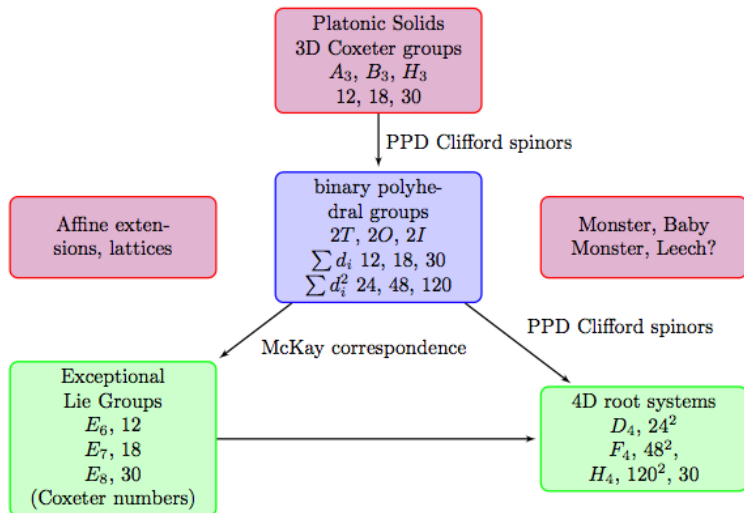
binary polyhedral groups
 $2T, 2O, 2I$
 $\sum d_i$ 12, 18, 30
 $\sum d_i^2$ 24, 48, 120

McKay correspondence

Exceptional Lie Groups
 $E_6, 12$
 $E_7, 18$
 $E_8, 30$
 (Coxeter numbers)



The McKay Correspondence



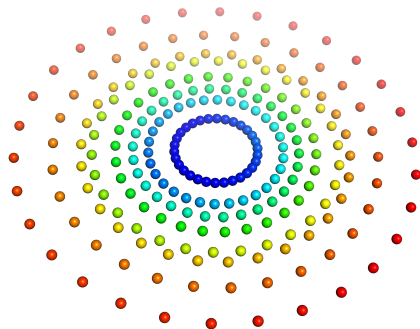
4D geometry is surprisingly important for HEP

- 4D root systems are **surprisingly relevant to HEP**
- A_4 is $SU(5)$ and comes up in **Grand Unification**
- D_4 is $SO(8)$ and is the little group of **String theory**
- In particular, its **triality symmetry** is crucial for showing the equivalence of RNS and GS strings
- B_4 is $SO(9)$ and is the little group of **M-Theory**
- F_4 is the **largest crystallographic** symmetry in 4D and H_4 is the **largest non-crystallographic** group
- The above are **subgroups** of the latter two
- **Spinorial nature** of the root systems could have **surprising consequences for HEP**

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Exceptional E_8 – the holy grail of maths and physics

- **Lie group** well-known (string theory, GUTs): triangular decomposition $248 = 120 + 8 + 120$
- **Root system** has **240** roots – 120 creation and annihilation operators, and 8 QN/Cartan degrees of freedom



Exceptional E_8 – from the icosahedron

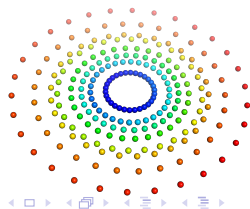
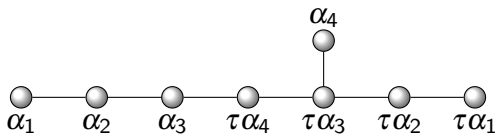
- Saw **even** products of the 30 roots of H_3 gave 120 **spinors** which in turn gave H_4 root system
- Taking **all** products gives group of 240 **pinors** with 8 components
- Essentially the inversion I just doubles the spinors

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1 e_2 e_3\}}_{1 \text{ trivector}}$$

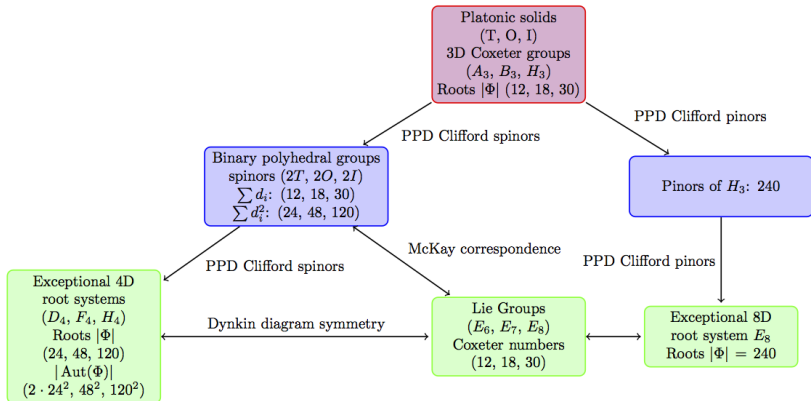
$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \quad \&IR = b_0 e_1 e_2 e_3 + b_1 e_1 + b_2 e_2 + b_3 e_3$$

- Most intuitive inner product on the pinors gives only $H_4 \oplus H_4$
- But slightly more technical **inner product** gives precisely the **E_8 root system** from the **icosahedron!**

- Order 120 group H_3 doubly covered by 240 (s)pinors
- Essentially $H_4 + IH_4$, two sets of 120
- Multiply second set by τI , take inner products, take into account $\tau^2 = \tau + 1$, but THEN: set $\tau \rightarrow 0!$ Each inner product is $(\alpha_i, \alpha_j) = a + \tau b \rightarrow (\alpha_i, \alpha_j)_\tau := a$
- Like the other exceptional geometries, E_8 is actually hidden within 3D geometry!

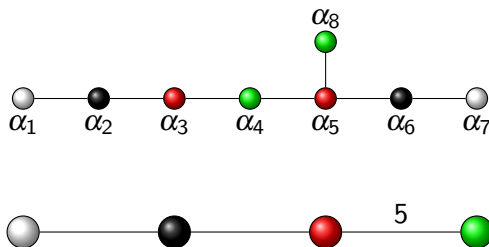


New, explicit connections – first examples of things requiring Clifford techniques?



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- 3 H_4 as a rotation group II: The Coxeter plane

Projection and Diagram Foldings

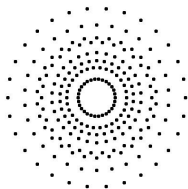
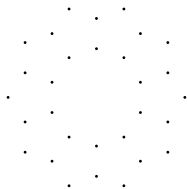
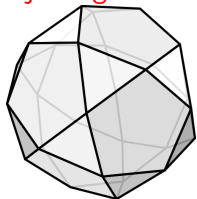


$$s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}, s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}, s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}, s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4$$

- E_8 has a H_4 subgroup of rotations via a 'partial folding'
- Can project 240 E_8 roots to $H_4 + \tau H_4$ – essentially the reverse of my construction!
- Coxeter element & number of E_8 and H_4 are the same

The Coxeter Plane

- Can show **every** (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by **projecting** their root system onto the Coxeter plane



Coxeter Elements, Degrees and Exponents

- Like the symmetric group, Coxeter groups can have **invariant polynomials**. Their **degrees** d are important invariants/group characteristics.
- Turns out that actually **degrees** d are intimately related to so-called **exponents** m $m = d - 1$.

Coxeter Elements, Degrees and Exponents

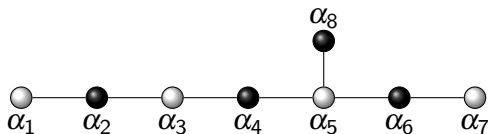
- A **Coxeter Element** is any combination of all the simple reflections $w = s_1 \dots s_n$, i.e. in Clifford algebra it is encoded by the versor $W = \alpha_1 \dots \alpha_n$ acting as $v \rightarrow wv = \pm \tilde{W}vW$. All such elements are conjugate and thus their **order** is invariant and called the **Coxeter number** h .
- The Coxeter element has **complex eigenvalues** of the form $\exp(2\pi mi/h)$ where m are called **exponents**:
 $wx = \exp(2\pi mi/h)x$
- Standard theory **complexifies** the real Coxeter group situation in order to find **complex eigenvalues**, then takes **real sections** again (the unfortunate standard procedure in many situations) – without any insight into the complex structure (or in fact, there are different ones).

Coxeter Elements, Degrees and Exponents

- The Coxeter element has **complex eigenvalues** of the form $\exp(2\pi mi/h)$ where m are called **exponents**
- Standard theory **complexifies** the real Coxeter group situation in order to find **complex eigenvalues**, then takes **real** sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure(s)
- In particular, **1** and **$h-1$** are always exponents
- Turns out that actually **exponents and degrees** are intimately related ($m = d - 1$). The construction is slightly roundabout but uniform, and uses the **Coxeter plane**.

The Coxeter Plane

- Obvious from Clifford point of view, that Coxeter element has eigenspaces (**eigenblades**) rather than just eigenvectors
- In particular, can show **every** (for our purposes) Coxeter group has a Coxeter plane
- Existence relies on the fact that all groups in question have **tree-like Dynkin diagrams**, and thus admit an **alternate colouring**
- Essentially just gives **two sets of mutually commuting generators**



The Coxeter Plane

- Existence relies on the fact that all groups in question have **tree-like Dynkin diagrams**, and thus admit an alternate colouring
- Essentially just gives **two sets of orthogonal = mutually commuting generators but anticommuting root vectors** α_w and α_b (duals ω)
- Cartan matrices are positive definite, and thus have a **Perron-Frobenius** (all positive) eigenvector λ_j .
- Take **linear combinations** of components of this eigenvector as coefficients of two vectors from the orthogonal sets
 $v_w = \sum \lambda_w \omega_w$ and $v_b = \sum \lambda_b \omega_b$
- Their **outer product/Coxeter plane bivector** $B_C = v_b \wedge v_w$ describes an **invariant plane** where w acts by rotation by $2\pi/h$.

Clifford Algebra and the Coxeter Plane – 2D case

$$I_2(n) \quad \circ \overset{n}{\text{---}} \circ$$

- For $I_2(n)$ take $\alpha_1 = e_1, \alpha_2 = -\cos \frac{\pi}{n} e_1 + \sin \frac{\pi}{n} e_2$
- So **Coxeter versor** is just

$$W = \alpha_1 \alpha_2 = -\cos \frac{\pi}{n} + \sin \frac{\pi}{n} e_1 e_2 = -\exp\left(-\frac{\pi I}{n}\right)$$

- In Clifford algebra it is therefore immediately obvious that the action of the $I_2(n)$ Coxeter element is described by a versor (here a rotor/spinor) that encodes **rotations in the $e_1 e_2$ -Coxeter-plane** and yields $h = n$ since trivially $W^n = (-1)^{n+1}$ yielding $w^n = 1$ via $wv = \tilde{W}vW$.

Clifford Algebra and the Coxeter Plane – 2D case

- So **Coxeter versor** is just $W = -\exp\left(-\frac{\pi I}{n}\right)$
- $I = e_1 e_2$ **anticommutes** with both e_1 and e_2 such that **sandwiching formula** becomes

$$v \rightarrow wv = \tilde{W}vW = \tilde{W}^2v = \exp\left(\pm\frac{2\pi I}{n}\right)v \text{ immediately}$$

yielding the standard result for the **complex eigenvalues** in real Clifford algebra **without any need for artificial complexification**

- The Coxeter plane bivector $B_C = e_1 e_2 = I$ gives the **complex structure**
- The Coxeter plane bivector B_C is invariant under the **Coxeter versor** $\tilde{W}B_CW = \pm B_C$.

Clifford Algebra and the Coxeter Plane – 3D case

- In 3D, A_3 , B_3 , H_3 have $\{1, 2, 3\}$, $\{1, 3, 5\}$ and $\{1, 5, 9\}$
- Coxeter element is product of a **spinor** in the Coxeter plane with the same complex structure as before, and a **reflection perpendicular** to the plane
- So in 3D still completely determined by the plane
- **1** and **$h-1$** are **rotations** in **Coxeter plane**
- **$h/2$** is the **reflection** (for v in the normal direction)

$$wv = \tilde{W}^2 = \exp\left(\pm \frac{2\pi I}{h} \frac{h}{2}\right) = \exp(\pm \pi I)v = -v$$

Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can **factorise** W into **orthogonal** (commuting/anticommuting) components

$$W = \alpha_1 \dots \alpha_n = W_1 \dots W_n \quad \text{with} \quad W_i = \exp(\pi m_i l_i / h)$$

- Here, l_i is a bivector describing a **plane** with $l_i^2 = -1$

- For v **orthogonal to the plane** described by l_i we have

$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i W_i v = v \quad \text{so cancels out}$$

- For v **in the plane** we have

$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i l_i / h) v$$

- Thus if we **decompose W into orthogonal eigenspaces**, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

Clifford algebra: no need for complexification

- For v in the plane we have

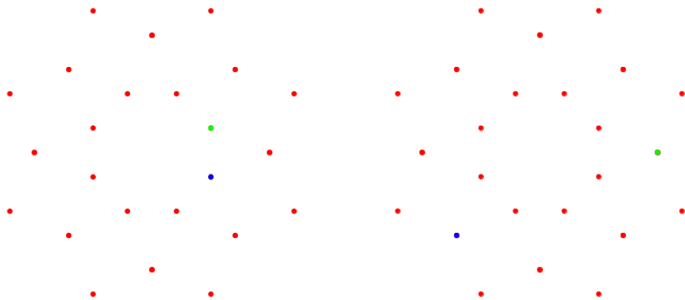
$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i / h) v$$

- So **complex eigenvalue equation** arises geometrically **without any need** for complexification
- **Different complex structures** immediately give different **eigenplanes**
- Eigenvalues/angles/**exponents** given from just factorising $W = \alpha_1 \dots \alpha_n$
- E.g. B_4 has exponents 1, 3, 5, 7 and $W = \exp\left(\frac{\pi}{8} I_1\right) \exp\left(\frac{3\pi}{8} I_2\right)$
- Here we have been looking for orthogonal eigenspaces, so **innocuous** – different complex structures commute
- But not in general – **naive complexification** can be misleading

4D case: B_4

- E.g. B_4 has exponents 1, 3, 5, 7
- Coxeter versor decomposes into **orthogonal components**

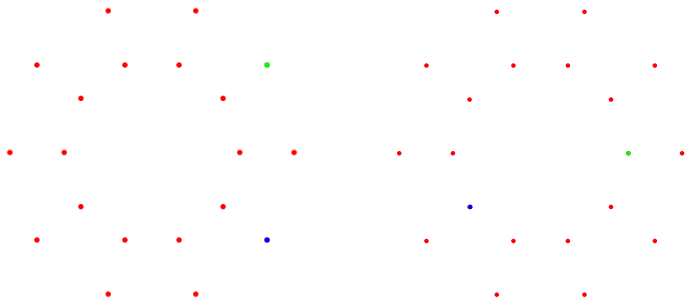
$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} I B_C\right)$$



4D case: A_4

- E.g. A_4 has exponents 1, 2, 3, 4
- Coxeter versor decomposes into **orthogonal components**

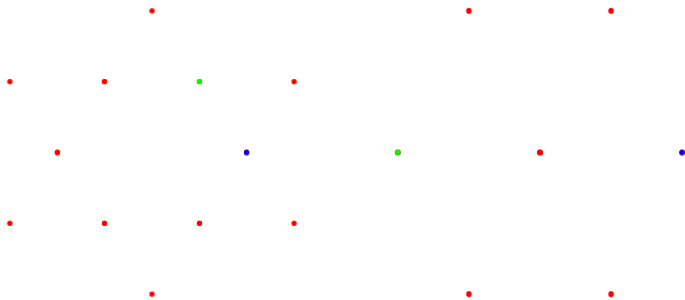
$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{5} B_C\right) \exp\left(\frac{2\pi}{5} I B_C\right)$$



4D case: D_4

- E.g. D_4 has exponents 1, 3, 3, 5
- Coxeter versor decomposes into **orthogonal components**

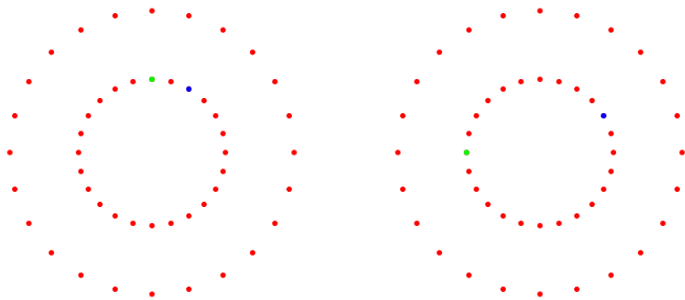
$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{3\pi}{6} I B_C\right)$$



4D case: F_4

- E.g. F_4 has exponents 1, 5, 7, 11
- Coxeter versor decomposes into **orthogonal components**

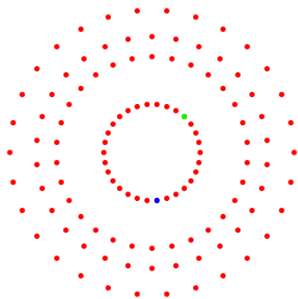
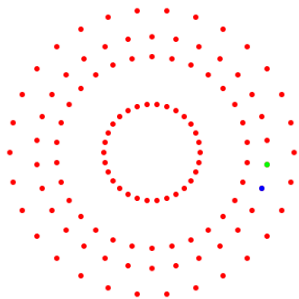
$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} I B_C\right)$$



4D case: H_4

- E.g. H_4 has exponents 1, 11, 19, 29
- Coxeter versor decomposes into **orthogonal components**

$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$$



Clifford Algebra and the Coxeter Plane – 4D case summary

rank 4	exponents	W-factorisation
A_4	1, 2, 3, 4	$W = \exp\left(\frac{\pi}{5} B_C\right) \exp\left(\frac{2\pi}{5} I B_C\right)$
B_4	1, 3, 5, 7	$W = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} I B_C\right)$
D_4	1, 3, 3, 5	$W = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{\pi}{2} I B_C\right)$
F_4	1, 5, 7, 11	$W = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} I B_C\right)$
H_4	1, 11, 19, 29	$W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$

Actually, in 2, 3 and 4 dimensions it couldn't really be any other way

Clifford Algebra and the Coxeter Plane – D_6

- For D_6 one has exponents $1, 3, 5, 5, 7, 9$
- Coxeter versor decomposes into orthogonal bits as

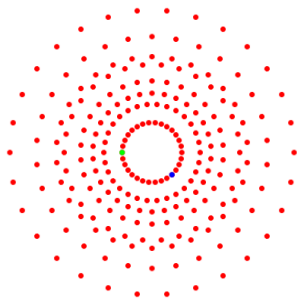
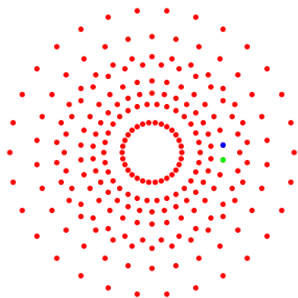
$$W = \frac{1}{\sqrt{5}}(e_1 + e_2 + e_3 - e_4 - e_5)e_6 \exp\left(\frac{\pi}{10}B_C\right) \exp\left(\frac{3\pi}{10}B_2\right)$$

- Now **bivector exponentials** correspond to **rotations in orthogonal planes**
- **Vector** factors correspond to **reflections**
- For odd n , there is always **one such vector factor** in D_n , and for even n there are **two**

8D case: E_8

- E.g. H_4 has exponents 1, 11, 19, 29, E_8 has 1, 7, 11, 13, 17, 19, 23, 29
- Coxeter versor decomposes into **orthogonal components**

$$W = \alpha_1 \dots \alpha_8 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{7\pi}{30} B_2\right) \exp\left(\frac{11\pi}{30} B_3\right) \exp\left(\frac{13\pi}{30} B_4\right)$$

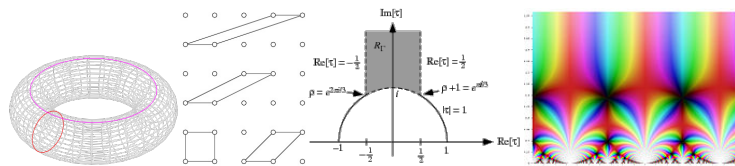


Imaginary differences – different imaginaries

So what has been **gained** by this **Clifford view**?

- There are **different** entities that serve as **unit imaginaries**
- They have a **geometric** interpretation as an **eigenplane of the Coxeter element**
- These don't need to **commute** with everything like i (though they do here – at least anticommute. But that is because we looked for **orthogonal decompositions**)
- But see that in general **naive complexification** can be a dangerous thing to do – **unnecessary**, issues of **commutativity**, **confusing** different imaginaries etc

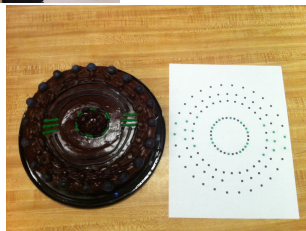
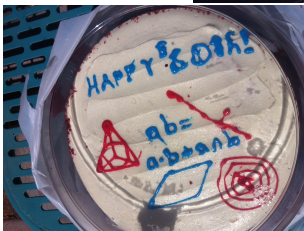
Modular group



- **Modular group**: interested in modular forms for applications in **Moonshine/string theory**: Monster 196883, Klein j 196884
- Modular generators: $T : \tau \rightarrow \tau + 1$, $S : \tau \rightarrow -1/\tau$
- $\langle S, T \mid S^2 = I, (ST)^3 = I \rangle$
- CGA: $T_X = 1 + \frac{ne_1}{2}$ and $S_X = e_1 e$
- $(S_X T_X)^3 = -1$ and $S_X^2 = 1$

Conclusions

- **All exceptional** geometries arise in **3D**, root systems giving rise to Lie groups/algebras etc
- Completely novel **spinorial** way of viewing the geometries as 3D phenomena – implications for HEP etc?
- More **natural** point of view, explaining **existence** and **automorphism groups**
- Totally unclear how one would see this in a **matrix framework** – might **require** Clifford point of view
- New view of Coxeter **degrees and exponents** with **geometric interpretation of imaginaries**
- A unified framework for doing **group and representation theory**: polyhedral, orthogonal, conformal, modular (Moonshine) etc



Thank you!
Congratulations David and Nancy!