The E_8 geometry from a Clifford perspective

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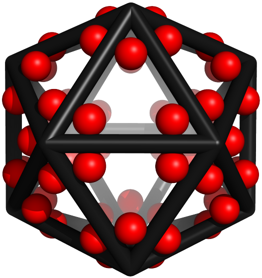
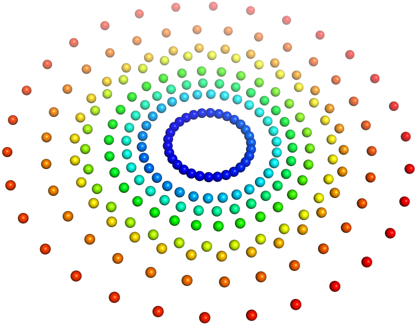
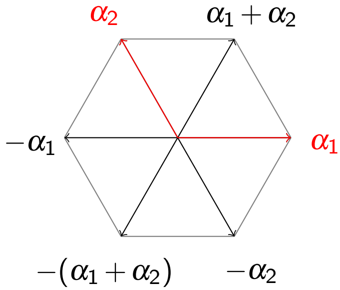
AGACSE 2012, La Rochelle: The birth of the E_8 question...

All exceptional geometries from 3D geometry





Root systems – A_2



Root system Φ: set of vectors α such that

1. $\boxed{\Phi\cap\mathbb{R}\alpha=\{-\alpha,\alpha\}\ \forall\,\alpha\in\Phi}$
2. $\boxed{s_\alpha\Phi=\Phi\ \forall\,\alpha\in\Phi}$

Simple roots: express every element of Φ via a $\mathbb{Z}$-linear combination (with coefficients of the same sign).

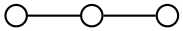
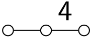
reflection/Coxeter groups $\boxed{s_\alpha: v\rightarrow s_\alpha(v)=v-2\frac{(v|\alpha)}{(\alpha|\alpha)}\alpha}$

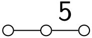
Cartan Matrices

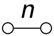
Cartan matrix of α_is is $\boxed{A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2\frac{|\alpha_j|}{|\alpha_i|}\cos\theta_{ij}}$

$$A_2\colon\ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Coxeter-Dynkin diagrams: node $=$ simple root, no link $=$ roots orthogonal, simple link $=$ roots at $\frac{\pi}{3}$, link with label m $=$ angle $\frac{\pi}{m}$.

A_3  B_3 

H_3 

$I_2(n)$ 

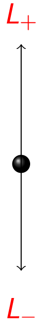
Coxeter groups vs Lie groups vs Lie algebras vs root systems

- Lie group = group and manifold (e.g. spin groups: Doran, Hestenes et al)
- Lie algebra = bilinear, antisymmetric bracket and Jacobi identity (e.g. bivector algebras) = Lie group near the identity
- 'Nice' Lie algebras have triangular decomposition:

$$\boxed{\mathscr{N}_- \oplus \mathscr{H} \oplus \mathscr{N}_+ : SU(2) : 1+1+1, E_8 : 120+8+120}$$

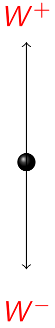
- $\mathscr{H}$: Cartan subalgebra/QN; Creation & annihilation ops $\mathscr{N}$: root lattice from crystallographic root systems
- Weyl group is a crystallographic Coxeter group: $\boxed{A_n, B_n/C_n, D_n, G_2, F_4, E_6, E_7, E_8}$.
- So via this route always crystallographic! Neglect $I_2(n), H_3, H_4$.

Example – A_1, $SU(2)$, Angular Momentum



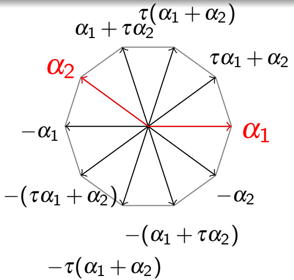
- Cartan subalgebra = Quantum number: L_z
- $\mathscr{N}_+$: raising operator $L_+ = \alpha$
- $\mathscr{N}_-$: lowering operator $L_- = -\alpha$
- (L^2 is Casimir/commutes with all algebra elements, is however not actually in the algebra!)

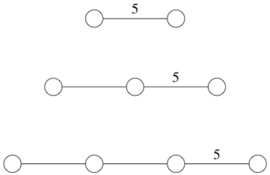
Example – A_1, $SU(2)$, Electroweak



- Cartan subalgebra – Quantum number: A
- $\mathcal{N}_+$: raising operator $W^+ = \alpha$
- $\mathcal{N}_-$: lowering operator $W^- = -\alpha$
- (Since SM electroweak is actually $SU(2) \times U(1)$, $U(1)$ gives another field i, such that physical Z^0 and γ are superpositions of A and i)
- Also $W^\pm$ now charged and self-interact, unlike QED

Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$





$$A=\begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}$$

$$A=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}$$

$$A=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

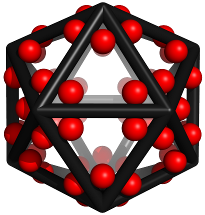
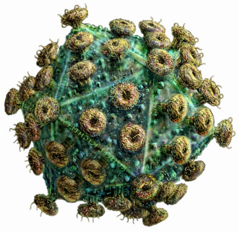
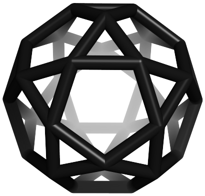
$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate rotational symmetries of order 5

linear combinations now in the extended integer ring

$\boxed{\mathbb{Z}[\tau]=\{a+\tau b | a,b\in\mathbb{Z}\}}$ golden ratio $\boxed{\tau=\frac{1}{2}(1+\sqrt{5})=2\cos\frac{\pi}{5}}$

$\boxed{x^2=x+1}$ $\boxed{\tau'=\sigma=\frac{1}{2}(1-\sqrt{5})=2\cos\frac{2\pi}{5}}$ $\boxed{\tau+\sigma=1, \tau\sigma=-1}$

The Icosahedron

- Rotational icosahedral group is $I = A_5$ of order 60
- Full icosahedral group is H_3 of order 120 (including reflections/inversion); generated by the root system icosidodecahedron

Clifford Algebra and orthogonal transformations

- Form an algebra using the Geometric Product for two vectors

$$\boxed{ab \equiv a \cdot b + a \wedge b}$$

- Inner product is symmetric $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting a in b is given by $\boxed{a' = a - 2(a \cdot b)b = -bab}$ (b and $-b$ doubly cover the same reflection)
- Via Cartan-Dieudonné theorem any orthogonal/conformal/modular transformation can be written as successive reflections

$$\boxed{x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1} = \pm A x \tilde{A}$$

Clifford Algebra of 3D

- E.g. Pauli algebra in 3D (likewise for Dirac algebra in 4D) is

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

- We can form the elements of the Coxeter groups by multiplying together root vectors in this algebra $\alpha_i\alpha_j\dots$
- General: 8 components, even products: (rotations/spinors) four components:

$$\boxed{R = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2 \Rightarrow R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

- So behaves as a 4D Euclidean object – inner product

$$\boxed{(R_1, R_2) = \frac{1}{2}(R_2\tilde{R}_1 + R_1\tilde{R}_2)}$$

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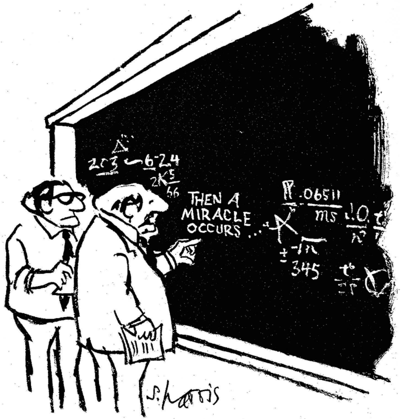
4D from 3D



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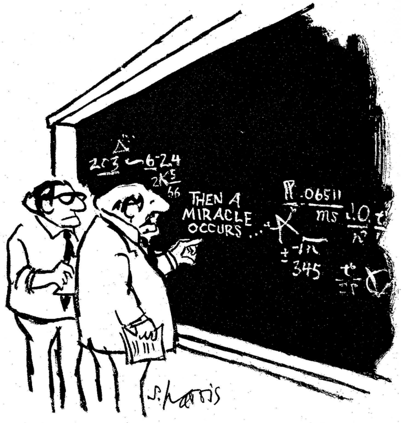
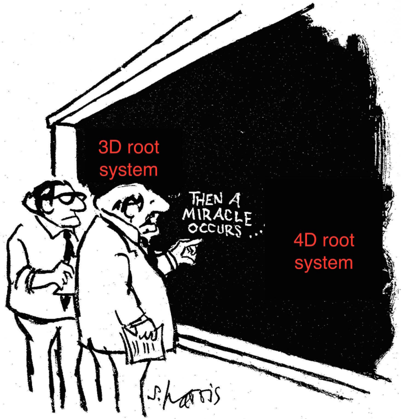
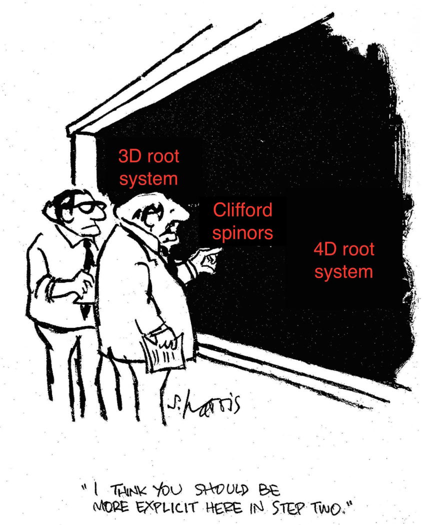


"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

4D from 3D



4D from 3D


3D root system
Clifford spinors
4D root system
S. Harris
"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Induction Theorem – root systems

- Theorem: 3D spinor groups give 4D root systems.

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- Check axioms:
  1. $\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall\, \alpha \in \Phi}$
  2. $\boxed{s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi}$

Induction Theorem – root systems

- Theorem: 3D spinor groups give 4D root systems.
- Check axioms:
  1. $\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall\, \alpha \in \Phi}$
  2. $\boxed{s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi}$
- Proof: 1. R and $-R$ are in a spinor group by construction (double cover of orthogonal transformations), 2. closure under reflections is guaranteed by the closure property of the spinor group (with a twist: $-R_1\tilde{R}_2R_1$)
- Induction Theorem: Every rank-3 root system induces a rank-4 root system (and thereby Coxeter groups)
- Counterexample: not every rank-4 root system is induced in this way

Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the Platonic Solids:
- The 6 reflections in $A_1 \times A_1 \times A_1$ generate 8 spinors.
- $\pm e_1$, $\pm e_2$, $\pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The discrete spinor group is isomorphic to the quaternion group Q.

Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the Platonic Solids:
- The 6/12/18/30 reflections in $A_1 \times A_1 \times A_1/A_3/B_3/H_3$ generate 8/24/48/120 spinors.
- E.g. $\pm e_1$, $\pm e_2$, $\pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The discrete spinor group is isomorphic to the quaternion group Q / binary tetrahedral group $2T$/ binary octahedral group $2O$/ binary icosahedral group $2I$).

A_1^3 A_3 B_3 H_3

A_1^4 D_4 F_4 H_4

Exceptional Root Systems

- The 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell are in fact the root systems of $A_1 \times A_1 \times A_1 \times A_1$, D_4, F_4 and H_4
- Exceptional phenomena: D_4 (triality, important in string theory), F_4 (largest lattice symmetry in 4D), H_4 (largest non-crystallographic symmetry)
- Exceptional D_4 and F_4 arise from series A_3 and B_3
- In fact, as we have seen one can strengthen this statement on inducing polytopes to a statement on inducing root systems

3D vs 4D

- Have A_n, B_n and D_n families of root systems in any dimension
- In 3D, have H_3 as an accident (icosahedron and dodecahedron)
- In 4D, have F_4 and H_4 (and in some sense D_4) as accidents
- These 4D accidents have unusual automorphism groups
- Can induce all of these from the 3D cases, show they are root systems and explain their automorphism groups

Root systems in three and four dimensions

The spinors generated from the reflections contained in the respective rank-3 Coxeter group via the geometric product are realisations of the binary polyhedral groups Q, $2T$, $2O$ and $2I$, which were known to generate (mostly exceptional) rank-4 groups, but not known why, and why the 'mysterious symmetries'.

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$	$\circ \quad \circ \quad \circ$	Q	$A_1 \times A_1 \times A_1 \times A_1$	$\circ \quad \circ \quad \circ \quad \circ$
A_3	$\circ - \circ - \circ$	$2T$	D_4	$\circ - \circ <^{\circ}_{\circ}$
B_3	$\circ - \circ \overset{4}{-} \circ$	$2O$	F_4	$\circ - \circ \overset{4}{-} \circ - \circ$
H_3	$\circ - \circ \overset{5}{-} \circ$	$2I$	H_4	$\circ - \circ - \circ \overset{5}{-} \circ$

Induction Theorem – automorphism

- So induced 4D polytopes are actually root systems.
- Clear why the number of roots $|\Phi|$ is equal to $|G|$, the order of the spinor group
- Spinor group is trivially closed under conjugation, left and right multiplication. Results in non-trivial symmetries when viewed as a polytope/root system.
- Now explains symmetry of the polytopes/root system and thus the order of the rank-4 Coxeter group
- Theorem: The automorphism group of the induced root system contains two factors of the respective spinor group acting from the left and the right.

Spinorial Symmetries of 4D Polytopes

Spinorial symmetries

rank 3	$\lvert\Phi\rvert$	$\lvert W\rvert$	rank 4	$\lvert\Phi\rvert$	Symmetry
A_3	12	24	D_4 24-cell	24	$2\cdot 24^2=576$
B_3	18	48	F_4 lattice	48	$48^2=2304$
H_3	30	120	H_4 600-cell	120	$120^2=14400$
A_1^3	6	8	A_1^4 16-cell	8	$3!\cdot 8^2=384$
$A_1\oplus A_2$	8	12	$A_2\oplus A_2$ prism	12	$12^2=144$
$A_1\oplus H_2$	12	20	$H_2\oplus H_2$ prism	20	$20^2=400$
$A_1\oplus I_2(n)$	$n+2$	$2n$	$I_2(n)\oplus I_2(n)$	$2n$	$(2n)^2$

Similar for Grand Antiprism (H_4 without $H_2\oplus H_2$) and Snub 24-cell ($2I$ without $2T$). Additional factors in the automorphism group come from 3D Dynkin diagram symmetries!

Arnold's Trinities

Arnold's observation that many areas of real mathematics can be complexified and quaternionified resulting in theories with a similar structure.

- The fundamental trinity is thus $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The projective spaces $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The spheres $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The Möbius/Hopf bundles $(S^1 \to S^1, S^4 \to S^2, S^7 \to S^4)$
- The Lie Algebras (E_6, E_7, E_8)
- The symmetries of the Platonic Solids (A_3, B_3, H_3)
- The 4D groups (D_4, F_4, H_4)
- New connections via my Clifford spinor construction (see McKay correspondence)

Platonic Trinities

- Arnold's connection between (A_3, B_3, H_3) and (D_4, F_4, H_4) is very convoluted and involves numerous other trinities at intermediate steps:
- Decomposition of the projective plane into Weyl chambers and Springer cones
- The number of Weyl chambers in each segment is $24 = 2(1+3+3+5), 48 = 2(1+5+7+11), 120 = 2(1+11+19+29)$
- Notice this miraculously matches the quasihomogeneous weights $((2,4,4,6),(2,6,8,12),(2,12,20,30))$ of the Coxeter groups (D_4, F_4, H_4)
- Believe the Clifford connection is more direct

A unified framework for polyhedral groups

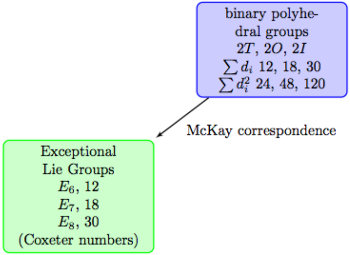
Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \to \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \to \pm\tilde{A}xA$
$\mathrm{Spin}(3)$	binary	$(R_1, R_2) \to R_1 R_2$
$\mathrm{Pin}(3)$	pinor	$(A_1, A_2) \to A_1 A_2$

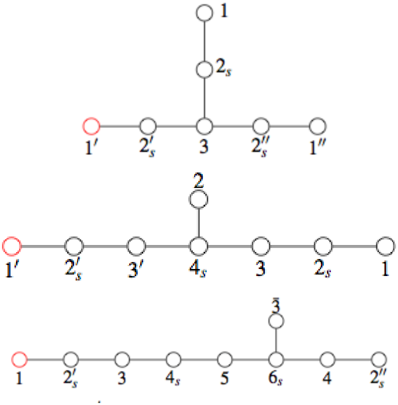
- e.g. the chiral icosahedral group has 60 elements, encoded in Clifford by 120 spinors, which form the binary icosahedral group
- together with the inversion/pseudoscalar I this gives 60 rotations and 60 rotoinversions, i.e. the full icosahedral group H_3 in 120 elements (with 240 pinors)
- all three are interesting groups, e.g. in neutrino and flavour physics for family symmetry model building

Some Group Theory: chiral, full, binary, pin

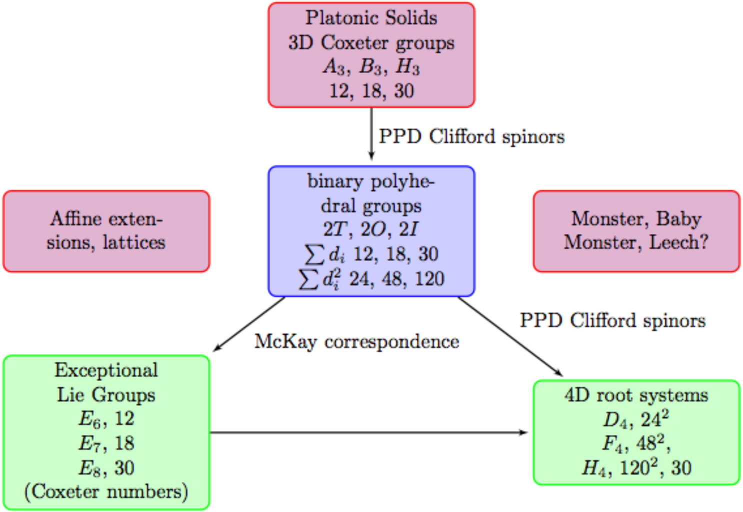
- Easy enough to calculate conjugacy classes etc of pinors in Clifford algebra
- Chiral (binary) polyhedral groups have irreps
- tetrahedral (12/24): 1, $1'$, $1''$, 2_s, $2'_s$, $2''_s$, 3
- octahedral (24/48): 1, $1'$, 2, 2_s, $2'_s$, 3, $3'$, 4_s
- icosahedral (60/120): 1, 2_s, $2'_s$, 3, $\bar{3}$, 4, 4_s, 5, 6_s
- Binary groups are discrete subgroups of $SU(2)$ and all thus have a 2_s spinor irrep
- Connection with the McKay correspondence!

The McKay Correspondence: Coxeter number, dimensions of irreps and (tensor product) graphs





The McKay Correspondence



4D geometry is surprisingly important for HEP

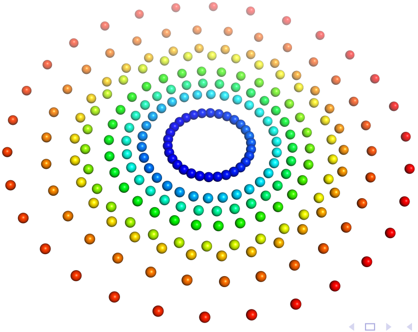
- 4D root systems are surprisingly relevant to HEP
- A_4 is $SU(5)$ and comes up in Grand Unification
- D_4 is $SO(8)$ and is the little group of String theory
- In particular, its triality symmetry is crucial for showing the equivalence of RNS and GS strings
- B_4 is $SO(9)$ and is the little group of M-Theory
- F_4 is the largest crystallographic symmetry in 4D and H_4 is the largest non-crystallographic group
- The above are subgroups of the latter two
- Spinorial nature of the root systems could have surprising consequences for HEP

Overview

1. H_4 as a rotation group I: spinor induction, Trinities and McKay correspondence
2. The birth of E_8 out of the spinors of the icosahedron
3. H_4 as a rotation group II: The Coxeter plane

Exceptional E_8 – the holy grail of maths and physics

- Lie group well-known (string theory, GUTs): triangular decomposition $248 = 120 + 8 + 120$
- Root system has 240 roots – 120 creation and annihilation operators, and 8 QN/Cartan degrees of freedom



Exceptional E_8 – from the icosahedron

- Saw even products of the 30 roots of H_3 gave 120 spinors which in turn gave H_4 root system
- Taking all products gives group of 240 pinors with 8 components
- Essentially the inversion I just doubles the spinors

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

$$\boxed{R = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2 \& IR = b_0e_1e_2e_3 + b_1e_1 + b_2e_2 + b_3e_3}$$

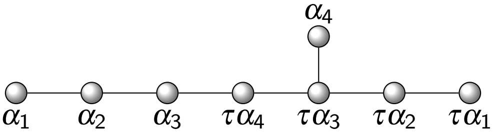
- Most intuitive inner product on the pinors gives only $H_4 \oplus H_4$
- But slightly more technical inner product gives precisely the E_8 root system from the icosahedron!

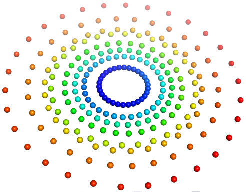
- Order 120 group H_3 doubly covered by 240 (s)pinors

- Essentially $H_4 + IH_4$, two sets of 120

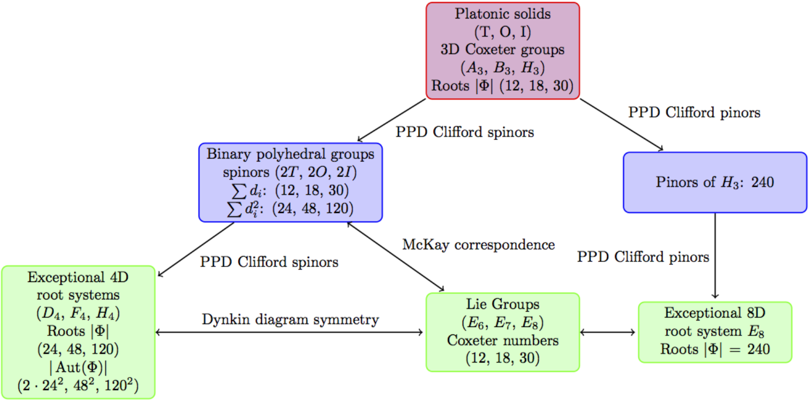
- Multiply second set by τI, take inner products, take into account $\tau^2 = \tau + 1$, but THEN: set $\tau \to 0$! Each inner product is $(\alpha_i, \alpha_j) = a + \tau b \to (\alpha_i, \alpha_j)_\tau := a$

- Like the other exceptional geometries, E_8 is actually hidden within 3D geometry!





New, explicit connections – first examples of things *requiring* Clifford techniques?



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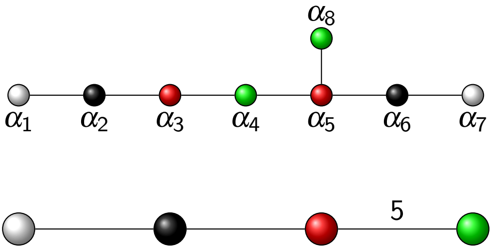
Projection and Diagram Foldings

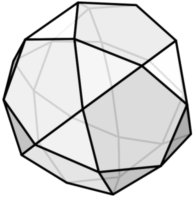


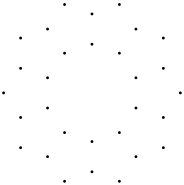
$$\boxed{s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}, \; s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}, \; s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}, \; s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4}$$

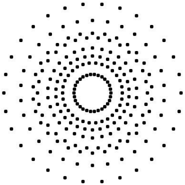
- E_8 has a H_4 subgroup of rotations via a 'partial folding'
- Can project 240 E_8 roots to $H_4 + \tau H_4$ – essentially the reverse of my construction!
- Coxeter element & number of E_8 and H_4 are the same

The Coxeter Plane

- Can show every (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by projecting their root system onto the Coxeter plane







Coxeter Elements, Degrees and Exponents

- Like the symmetric group, Coxeter groups can have invariant polynomials. Their degrees d are important invariants/group characteristics.
- Turns out that actually degrees d are intimately related to so-called exponents m $\boxed{m = d - 1}$.

Coxeter Elements, Degrees and Exponents

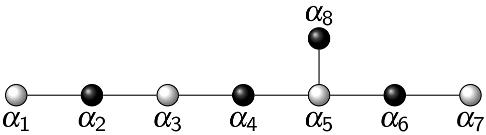
- A Coxeter Element is any combination of all the simple reflections $\boxed{w = s_1 \dots s_n}$, i.e. in Clifford algebra it is encoded by the versor $W = \alpha_1 \dots \alpha_n$ acting as $\boxed{v \to wv = \pm \tilde{W} v W}$. All such elements are conjugate and thus their order is invariant and called the Coxeter number h.
- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where m are called exponents: $\boxed{wx = \exp(2\pi mi/h)x}$
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure (or in fact, there are different ones).

Coxeter Elements, Degrees and Exponents

- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where m are called exponents
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure(s)
- In particular, 1 and $h-1$ are always exponents
- Turns out that actually exponents and degrees are intimately related ($\boxed{m=d-1}$). The construction is slightly roundabout but uniform, and uses the Coxeter plane.

The Coxeter Plane

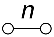
- Obvious from Clifford point of view, that Coxeter element has eigenspaces (eigenblades) rather than just eigenvectors
- In particular, can show every (for our purposes) Coxeter group has a Coxeter plane
- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of mutually commuting generators



The Coxeter Plane

- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of orthogonal = mutually commuting generators but anticommuting root vectors α_w and α_b (duals ω)
- Cartan matrices are positive definite, and thus have a Perron-Frobenius (all positive) eigenvector λ_i.
- Take linear combinations of components of this eigenvector as coefficients of two vectors from the orthogonal sets $v_w = \sum \lambda_w \omega_w$ and $v_b = \sum \lambda_b \omega_b$
- Their outer product/Coxeter plane bivector $\boxed{B_C = v_b \wedge v_w}$ describes an invariant plane where w acts by rotation by $2\pi/h$.

Clifford Algebra and the Coxeter Plane – 2D case

$I_2(n)$ 

- For $I_2(n)$ take $\boxed{\alpha_1 = e_1,\ \alpha_2 = -\cos\frac{\pi}{n}e_1 + \sin\frac{\pi}{n}e_2}$
- So Coxeter versor is just

$$\boxed{W = \alpha_1\alpha_2 = -\cos\frac{\pi}{n} + \sin\frac{\pi}{n}e_1e_2 = -\exp\left(-\frac{\pi I}{n}\right)}$$

- In Clifford algebra it is therefore immediately obvious that the action of the $I_2(n)$ Coxeter element is described by a versor (here a rotor/spinor) that encodes rotations in the e_1e_2-Coxeter-plane and yields $h = n$ since trivially $W^n = (-1)^{n+1}$ yielding $\boxed{w^n = 1}$ via $wv = \tilde{W}vW$.

Clifford Algebra and the Coxeter Plane – 2D case

- So Coxeter versor is just $\boxed{W = -\exp\left(-\frac{\pi I}{n}\right)}$
- $\boxed{I = e_1 e_2}$ anticommutes with both e_1 and e_2 such that sandwiching formula becomes
$$\boxed{v \rightarrow wv = \tilde{W} v W = \tilde{W}^2 v = \exp\left(\pm\frac{2\pi I}{n}\right) v}$$ immediately yielding the standard result for the complex eigenvalues in real Clifford algebra without any need for artificial complexification
- The Coxeter plane bivector $B_C = e_1 e_2 = I$ gives the complex structure
- The Coxeter plane bivector B_C is invariant under the Coxeter versor $\tilde{W} B_C W = \pm B_C$.

Clifford Algebra and the Coxeter Plane – 3D case

- In 3D, A_3, B_3, H_3 have $\{1,2,3\}$, $\{1,3,5\}$ and $\{1,5,9\}$
- Coxeter element is product of a spinor in the Coxeter plane with the same complex structure as before, and a reflection perpendicular to the plane
- So in 3D still completely determined by the plane
- 1 and $h-1$ are rotations in Coxeter plane
- $h/2$ is the reflection (for v in the normal direction)

$$\boxed{wv = \tilde{W}^2 = \exp(\pm\frac{2\pi I}{h}\frac{h}{2}) = \exp(\pm\pi I)v = -v}$$

Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can factorise W into orthogonal (commuting/anticommuting) components
  $\boxed{W = \alpha_1 \dots \alpha_n = W_1 \dots W_n}$ with $\boxed{W_i = \exp(\pi m_i I_i / h)}$
- Here, I_i is a bivector describing a plane with $\boxed{I_i^2 = -1}$
- For v orthogonal to the plane descrbed by I_i we have
  $\boxed{v \to \tilde{W}_i v W_i = \tilde{W}_i W_i v = v}$ so cancels out
- For v in the plane we have
  $\boxed{v \to \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i / h) v}$
- Thus if we decompose W into orthogonal eigenspaces, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

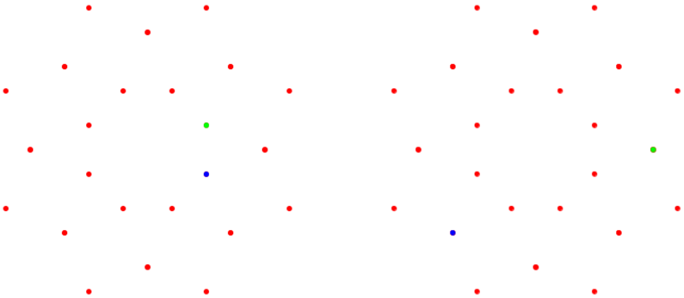
Clifford algebra: no need for complexification

- For v in the plane we have
  $$\boxed{v \to \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i / h) v}$$
- So complex eigenvalue equation arises geometrically without any need for complexification
- Different complex structures immediately give different eigenplanes
- Eigenvalues/angles/exponents given from just factorising $W = \alpha_1 \dots \alpha_n$
- E.g. B_4 has exponents $1, 3, 5, 7$ and $W = \exp\left(\frac{\pi}{8} I_1\right) \exp\left(\frac{3\pi}{8} I_2\right)$
- Here we have been looking for orthogonal eigenspaces, so innocuous – different complex structures commute
- But not in general – naive complexification can be misleading

4D case: B_4

- E.g. B_4 has exponents $1, 3, 5, 7$
- Coxeter versor decomposes into orthogonal components

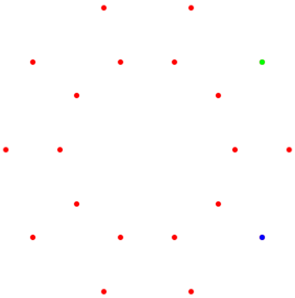
$$\boxed{W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{8}B_C\right)\exp\left(\frac{3\pi}{8}IB_C\right)}$$

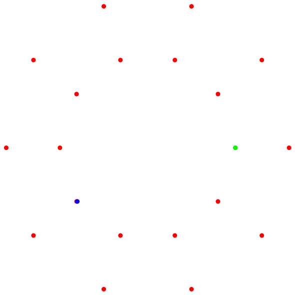


4D case: A_4

- E.g. A_4 has exponents $1, 2, 3, 4$
- Coxeter versor decomposes into orthogonal components

$$\boxed{W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{5}B_C\right)\exp\left(\frac{2\pi}{5}IB_C\right)}$$

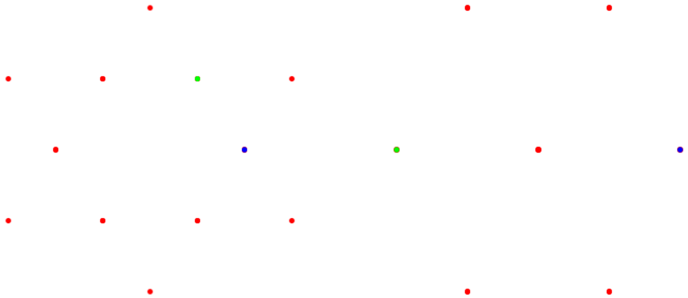




4D case: D_4

- E.g. D_4 has exponents $1, 3, 3, 5$
- Coxeter versor decomposes into orthogonal components

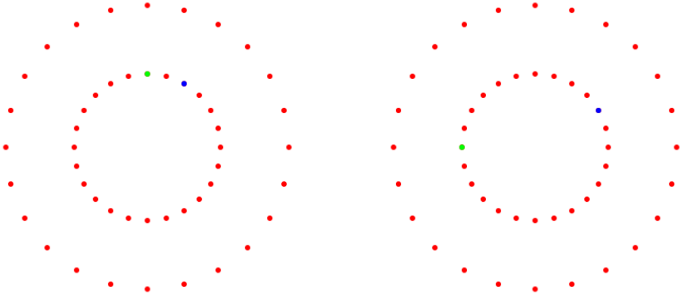
$$\boxed{W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{6}B_C\right)\exp\left(\frac{3\pi}{6}IB_C\right)}$$



4D case: F_4

- E.g. F_4 has exponents $1, 5, 7, 11$
- Coxeter versor decomposes into orthogonal components

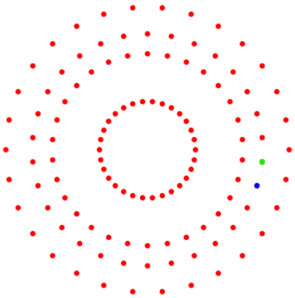
$$\boxed{W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{12}B_C\right)\exp\left(\frac{5\pi}{12}IB_C\right)}$$

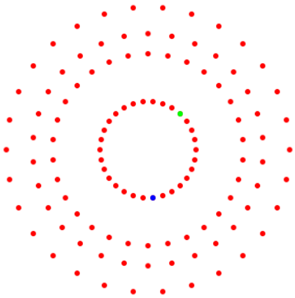


4D case: H_4

- E.g. H_4 has exponents $1, 11, 19, 29$
- Coxeter versor decomposes into orthogonal components

$$\boxed{W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{11\pi}{30}IB_C\right)}$$





Clifford Algebra and the Coxeter Plane – 4D case summary

rank 4	exponents	W-factorisation
A_4	$1, 2, 3, 4$	$W = \exp\left(\frac{\pi}{5} B_C\right) \exp\left(\frac{2\pi}{5} I B_C\right)$
B_4	$1, 3, 5, 7$	$W = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} I B_C\right)$
D_4	$1, 3, 3, 5$	$W = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{\pi}{2} I B_C\right)$
F_4	$1, 5, 7, 11$	$W = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} I B_C\right)$
H_4	$1, 11, 19, 29$	$W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$

Actually, in 2, 3 and 4 dimensions it couldn't really be any other way

Clifford Algebra and the Coxeter Plane – D_6

- For D_6 one has exponents $\boxed{1, 3, 5, 5, 7, 9}$
- Coxeter versor decomposes into orthogonal bits as

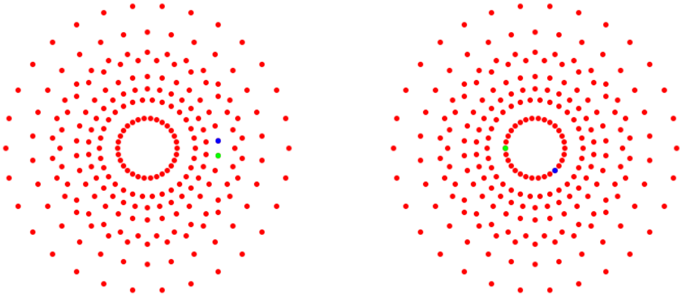
$$\boxed{W = \frac{1}{\sqrt{5}}(e_1 + e_2 + e_3 - e_4 - e_5)e_6 \exp\left(\frac{\pi}{10}B_C\right) \exp\left(\frac{3\pi}{10}B_2\right)}$$

- Now bivector exponentials correspond to rotations in orthogonal planes
- Vector factors correspond to reflections
- For odd n, there is always one such vector factor in D_n, and for even n there are two

8D case: E_8

- E.g. H_4 has exponents $1, 11, 19, 29$, E_8 has $1, 7, 11, 13, 17, 19, 23, 29$
- Coxeter versor decomposes into orthogonal components

$$\boxed{W = \alpha_1 \dots \alpha_8 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{7\pi}{30} B_2\right) \exp\left(\frac{11\pi}{30} B_3\right) \exp\left(\frac{13\pi}{30} B_4\right)}$$

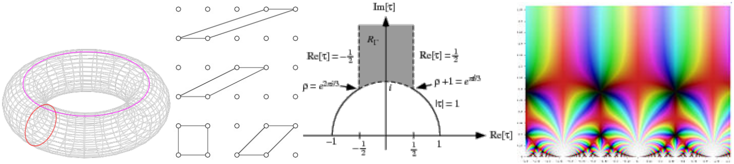


Imaginary differences – different imaginaries

So what has been gained by this Clifford view?

- There are different entities that serve as unit imaginaries
- They have a geometric interpretation as an eigenplane of the Coxeter element
- These don't need to commute with everything like i (though they do here – at least anticommute. But that is because we looked for orthogonal decompositions)
- But see that in general naive complexification can be a dangerous thing to do – unnecessary, issues of commutativity, confusing different imaginaries etc

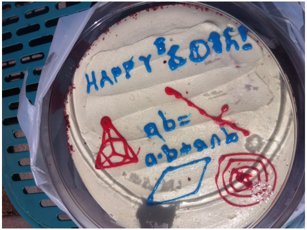
Modular group

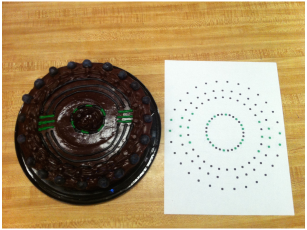


- Modular group: interested in modular forms for applications in Moonshine/string theory: Monster 196883, Klein j 196884
- Modular generators: $T: \tau \to \tau + 1$, $S: \tau \to -1/\tau$
- $\boxed{\langle S, T | S^2 = I, (ST)^3 = I \rangle}$
- CGA: $T_X = 1 + \frac{ne_1}{2}$ and $S_X = e_1 e$
- $(S_X T_X)^3 = -1$ and $S_X^2 = 1$

Conclusions

- **All** exceptional geometries arise in 3D, root systems giving rise to Lie groups/algebras etc
- Completely novel spinorial way of viewing the geometries as 3D phenomena – implications for HEP etc?
- More natural point of view, explaining existence and automorphism groups
- Totally unclear how one would see this in a matrix framework – might require Clifford point of view
- New view of Coxeter degrees and exponents with geometric interpretation of imaginaries
- A unified framework for doing group and representation theory: polyhedral, orthogonal, conformal, modular (Moonshine) etc





Thank you!
Congratulations David and Nancy!