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Platonic solids generate their four-dimensional analogues – a 3D spinorial view of 4D exceptional phenomena

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Birkbeck – November 10, 2014

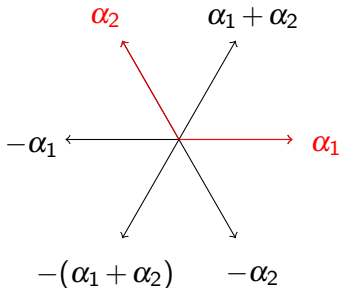
## 1 Introduction

- Coxeter groups and root systems
- Clifford algebras
- 'Platonic' Solids

## 2 Combining Coxeter and Clifford

- The Induction Theorem – from 3D to 4D
- Automorphism Groups
- Trinities and McKay correspondence

## Root systems – $A_2$



**Root system**  $\Phi$ : set of vectors  $\alpha$  such that

- $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$
- $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

**Simple roots**: express every element of  $\Phi$  via a  **$\mathbb{Z}$ -linear combination** (with coefficients of the same sign).

# Coxeter groups

A **Coxeter group** is a group generated by some **involutive generators**  $s_i, s_j \in S$  subject to relations of the form  $(s_i s_j)^{m_{ij}} = 1$  with  $m_{ij} = m_{ji} \geq 2$  for  $i \neq j$ .

# Coxeter groups

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The **finite** Coxeter groups have a **geometric representation** where the involutions are realised as **reflections at hyperplanes through the origin** in a Euclidean vector space  $\mathcal{E}$ . In particular, let  $(\cdot|\cdot)$  denote the inner product in  $\mathcal{E}$ , and  $v, \alpha \in \mathcal{E}$ .

The **generator**  $s_\alpha$  corresponds to the **reflection**

$$s_\alpha : v \rightarrow s_\alpha(v) = v - 2 \frac{(v|\alpha)}{(\alpha|\alpha)} \alpha$$

at a hyperplane perpendicular to the **root vector**  $\alpha$ .

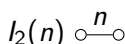
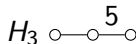
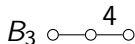
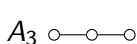
The action of the **Coxeter group** is to permute these **root vectors**.

# Cartan Matrices

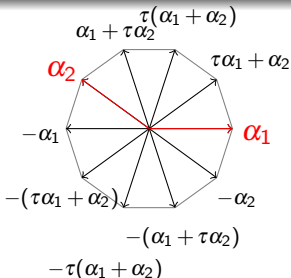
Cartan matrix of  $\alpha_i$ s is  $A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2 \frac{|\alpha_j|}{|\alpha_i|} \cos \theta_{ij}$

$$A_2: A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

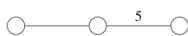
Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal, simple link = roots at  $\frac{\pi}{3}$ , link with label  $m =$  angle  $\frac{\pi}{m}$ .



# Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$



$$A = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$ : 10, 120, 14,400 elements, the only Coxeter groups that generate **rotational symmetries of order 5** linear combinations now in the **extended integer ring**

$$\mathbb{Z}[\tau] = \{a + \tau b \mid a, b \in \mathbb{Z}\} \quad \text{golden ratio} \quad \tau = \frac{1}{2}(1 + \sqrt{5}) = 2 \cos \frac{\pi}{5}$$

$$x^2 = x + 1 \quad \tau' = \sigma = \frac{1}{2}(1 - \sqrt{5}) = 2 \cos \frac{2\pi}{5} \quad \tau + \sigma = 1, \tau\sigma = -1$$

# Basics of Clifford Algebra I

- Form an algebra using the **Geometric Product**

$$ab \equiv a \cdot b + a \wedge b \text{ for two vectors}$$

- Extend via linearity and associativity to higher grade elements (**multivectors**)
- For an  $n$ -dimensional space generated by  $n$  orthogonal unit vectors  $e_i$  have  $2^n$  elements
- Then  $e_i e_j = e_i \wedge e_j = -e_j e_i$  so **anticommute** (Grassmann variables, exterior algebra)
- Unlike the **inner** and **outer** products separately, this product is **invertible**

## Basics of Clifford Algebra II

- These are known to have **matrix representations** over the normed division algebras  $\mathbb{R}$ ,  $\mathbb{C}$  and  $\mathbb{H}$   $\Rightarrow$  **Classification** of Clifford algebras
- E.g. **Pauli algebra** in 3D (likewise for **Dirac algebra** in 4D) is

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1 e_2 e_3\}}_{1 \text{ trivector}}$$

- These have the well-known matrix representations in terms of  **$\sigma$ - and  $\gamma$ -matrices**
- Working with these is not necessarily the most insightful thing to do, so here stress approach to **work directly** with the algebra

# Reflections

- Clifford algebra is **very efficient** at performing **reflections**
- Consider reflecting the vector  $a = a_{\perp} + a_{\parallel}$  in a hypersurface with unit normal  $n$ :

$$a' = a_{\perp} - a_{\parallel} = a - 2a_{\parallel} = a - 2(a \cdot n)n$$

- c.f. **fundamental Weyl reflection**  $s_i : v \rightarrow s_i(v) = v - 2 \frac{(v|\alpha_i)}{(\alpha_i|\alpha_i)} \alpha_i$
- But in Clifford algebra have  $n \cdot a = \frac{1}{2}(na + an)$  so reassembles into (note doubly covered by  $n$  and  $-n$ ) **sandwiching**

$$a' = -nan$$

- So both **Coxeter** and **Clifford** frameworks are ideally suited to describing **reflections** – combine the two

# Rotations

- Generate a **rotation** in the plane  $m \wedge n$  when compounding two reflections wrt  $n$  then  $m$ :

$$a'' = mnanm \equiv Ra\tilde{R}$$

where  $R = mn$  is called a **rotor** and a tilde denotes **reversal** of the order of the constituent vectors ( $R\tilde{R} = 1$ )

- Multivectors transform **covariantly** e.g.

$$MN \rightarrow (RM\tilde{R})(RN\tilde{R}) = RM\tilde{R}RN\tilde{R} = R(MN)\tilde{R}$$

so transform **double-sidedly**

- Spinors form a **group**, which gives a representation of the **Spin group**  $Spin(n)$  – they transform **single-sidedly** (obvious it's a double (universal) cover)

# Geometric Algebra and orthogonal transformations

- **Cartan-Dieudonné**: every isometry is at most  $d$  reflections
- Since have a **double cover** of reflections ( $n$  and  $-n$ ) we have a **double cover** of  $O(p, q)$ : **Pin**( $p, q$ )

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1$$

- **Pinors** = products of vectors  $n_1 n_2 \dots n_k$  encode orthogonal transformations via '**sandwiching**'
- **Cartan-Dieudonné**: rotations are an **even** number of reflections: **Spin**( $p, q$ ) doubly covers  $SO(p, q)$

# 3D Platonic Solids



- There are 5 Platonic solids
- Tetrahedron (**self-dual**) ( $A_3$ )
- **Dual** pair **octahedron** and **cube** ( $B_3$ )
- **Dual** pair **icosahedron** and **dodecahedron** ( $H_3$ )
- Only the **octahedron** is a **root system** (actually for  $(A_1^3)$ )

# Clifford and Coxeter: Platonic Solids



$A_1^3$

$A_3$

$B_3$

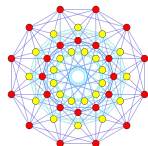
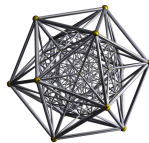
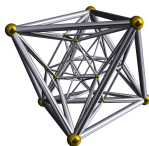
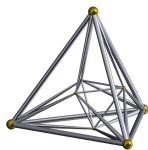
$H_3$

Platonic Solid	Group	root system
Tetrahedron	$A_3$ $A_1^3$	Cuboctahedron <b>Octahedron</b>
Octahedron Cube	$B_3$	Cuboctahedron + Octahedron
Icosahedron Dodecahedron	$H_3$	Icosidodecahedron

- **Platonic Solids** have been known for millennia
- Described by **Coxeter** groups

## 4D 'Platonic Solids'

- In 4D, there are **6 analogues** of the Platonic Solids:
- **5-cell** (self-dual) ( $A_4$ )
- **24-cell** (self-dual) ( $D_4$ ) – a 24-cell and its dual together are the  $F_4$  root system
- Dual pair **16-cell** and **8-cell** ( $B_4$ )
- Dual pair **600-cell** and **120-cell** ( $H_4$ )
- These are 4D analogues of the **Platonic Solids**: regular convex 4-polytopes



## 4D 'Platonic Solids'

- 24-cell, 16-cell and 600-cell are all root systems, as is the related  $F_4$  root system
- 8-cell and 120-cell are dual to a root system, so in 4D out of 6 Platonic Solids only the 5-cell (corresponding to  $A_n$  family) is not related to a root system!
- The 4D Platonic solids are not normally thought to be related to the 3D ones except for the boundary cells
- They have very unusual automorphism groups
- Some partial case-by-case algebraic results in terms of quaternions – here we show a uniform construction offering geometric understanding

# Mysterious Symmetries of 4D Polytopes

## Spinorial symmetries

rank 4	$ \Phi $	Symmetry
$D_4$ 24-cell	24	$2 \cdot 24^2 = 576$
$F_4$ lattice	48	$48^2 = 2304$
$H_4$ 600-cell	120	$120^2 = 14400$
$A_1^4$ 16-cell	8	$3! \cdot 8^2 = 384$
$A_2 \oplus A_2$ prism	12	$12^2 = 144$
$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

Similar for **Grand Antiprism** ( $H_4$  without  $H_2 \oplus H_2$ ) and **Snub 24-cell** ( $2I$  without  $2T$ ).

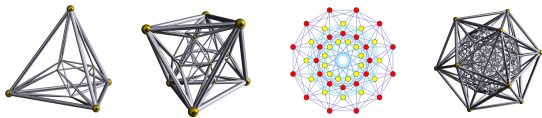
# A new connection



$A_1^3$	$A_1^4$
$A_3$	$D_4$
$B_3$	$F_4$
$H_3$	$H_4$

- **Platonic Solids** have been known for millennia; described by **Coxeter** groups
- Concatenating reflections gives **Clifford** spinors (**binary polyhedral groups**)
- These **induce 4D root systems**  

$$\psi = a_0 + a_i |e_i \Rightarrow \psi \tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$
- 4D analogues of the Platonic Solids and give rise to 4D **Coxeter** groups



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## 2 Combining Coxeter and Clifford

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# Induction Theorem – root systems

- Theorem: **3D spinor groups** give **root systems**.
  1.  $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$
  2.  $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$
- Proof: 1.  **$R$  and  $-R$**  are in a spinor group by construction, 2. closure under reflections is guaranteed by the **closure property of the spinor group**
- Induction Theorem: **Every rank-3 root system induces a rank-4 root system** (and thereby **Coxeter groups**)
- Counterexample: **not every rank-4 root system** is induced in this way

# Induction Theorem – automorphism

- So induced **4D polytopes** are actually **root systems**.
- Clear why the **number of roots**  $|\Phi|$  is equal to  $|G|$ , the **order of the spinor group**
- Spinor group is trivially **closed** under **conjugation, left and right multiplication**. Results in **non-trivial symmetries** when viewed as a **polytope/root system**.
- Now explains **symmetry** of the polytopes/root system and thus the **order** of the rank-4 Coxeter group
- Theorem: The **automorphism group** of the induced root system contains **two factors** of the respective spinor group acting from the **left** and the **right**.

# Recap: Clifford algebra and reflections & rotations

- Clifford algebra is **very efficient** at performing **reflections** via **sandwiching**

$$a' = -nan$$

- Generate a **rotation** when compounding two reflections wrt  $n$  then  $m$  (**Cartan-Dieudonné theorem**):

$$a'' = mnanm \equiv Ra\tilde{R}$$

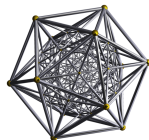
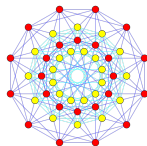
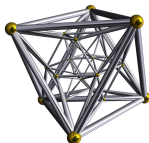
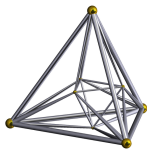
where  $R = mn$  is called a **spinor** and a tilde denotes **reversal** of the order of the constituent vectors ( $R\tilde{R} = 1$ )

## Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the **Platonic Solids**:
- The 6/12/18/30 **reflections** in  $A_1 \times A_1 \times A_1 / A_3 / B_3 / H_3$  generate 8/24/48/120 **spinors**.
- E.g.  $\pm e_1, \pm e_2, \pm e_3$  give the 8 spinors  $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The **discrete spinor group** is isomorphic to the **quaternion group  $Q$**  / **binary tetrahedral group  $2T$**  / **binary octahedral group  $2O$**  / **binary icosahedral group  $2I$** ).

# Spinors and Polytopes

- The space of  $Cl(3)$ -spinors and quaternions have a **4D Euclidean signature**:  $\psi = a_0 + a_i |e_i \Rightarrow \psi \tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- Can reinterpret **spinors in  $\mathbb{R}^3$**  as **vectors in  $\mathbb{R}^4$**
- Then the spinors constitute the **vertices** of the **16-cell**, **24-cell**, **24-cell and dual 24-cell** and the **600-cell**
- These are 4D analogues of the **Platonic Solids**: regular convex 4-polytopes






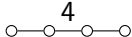

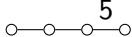


# Exceptional Root Systems

- The 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell are in fact the root systems of  $A_1 \times A_1 \times A_1 \times A_1$ ,  $D_4$ ,  $F_4$  and  $H_4$
- Exceptional phenomena:  $D_4$  (triality, important in string theory),  $F_4$  (largest lattice symmetry in 4D),  $H_4$  (largest non-crystallographic symmetry)
- Exceptional  $D_4$  and  $F_4$  arise from series  $A_3$  and  $B_3$
- In fact, as we have seen one can strengthen this statement on inducing polytopes to a statement on inducing root systems

# Root systems in three and four dimensions

The **spinors** generated from the reflections contained in the respective **rank-3 Coxeter group** via the geometric product are realisations of the **binary polyhedral groups**  $Q$ ,  $2T$ ,  $2O$  and  $2I$ , which were known to generate (mostly exceptional) **rank-4 groups**, but **not known why**, and why the ‘**mysterious symmetries**’.

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$		$Q$	$A_1 \times A_1 \times A_1 \times A_1$	
$A_3$		$2T$	$D_4$	
$B_3$		$2O$	$F_4$	
$H_3$		$2I$	$H_4$	

## General Case of Induction

Only **remaining case** is what happens for  $A_1 \oplus I_2(n)$  - this gives a **doubling**  $I_2(n) \oplus I_2(n)$

rank 3	rank 4
$A_3$	$D_4$
$B_3$	$F_4$
$H_3$	$H_4$
$A_1^3$	$A_1^4$
$A_1 \oplus A_2$	$A_2 \oplus A_2$
$A_1 \oplus H_2$	$H_2 \oplus H_2$
$A_1 \oplus I_2(n)$	$I_2(n) \oplus I_2(n)$

Can do an analogous construction using 3 roots to generate a discrete **octonion group**. These are again **root systems**, however just **two copies of the above**.

# Automorphism Groups

- So induced **4D polytopes** are actually **root systems** via the binary polyhedral groups.
- Clear why the **number of roots**  $|\Phi|$  is equal to  $|G|$ , the **order of the spinor group**.
- Spinor group is trivially **closed** under **conjugation, left and right multiplication**. Results in **non-trivial symmetries** when viewed as a **polytope/root system**.
- Now explains **symmetry** of the polytopes/root system and thus the **order** of the rank-4 Coxeter group
- Theorem: The **automorphism group** of the induced root system contains **two factors** of the respective spinor group acting from the **left** and the **right**.

## Spinorial Symmetries of 4D Polytopes

## Spinorial symmetries

rank 3	$ \Phi $	$ W $	rank 4	$ \Phi $	Symmetry
$A_3$	12	24	$D_4$ 24-cell	24	$2 \cdot 24^2 = 576$
$B_3$	18	48	$F_4$ lattice	48	$48^2 = 2304$
$H_3$	30	120	$H_4$ 600-cell	120	$120^2 = 14400$
$A_1^3$	6	8	$A_1^4$ 16-cell	8	$3! \cdot 8^2 = 384$
$A_1 \oplus A_2$	8	12	$A_2 \oplus A_2$ prism	12	$12^2 = 144$
$A_1 \oplus H_2$	12	20	$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$A_1 \oplus I_2(n)$	$n+2$	$2n$	$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

Similar for **Grand Antiprism** ( $H_4$  without  $H_2 \oplus H_2$ ) and **Snub 24-cell** ( $2I$  without  $2T$ ). Additional factors in the automorphism group come from **3D Dynkin diagram symmetries!**

## Some non-Platonic examples of spinorial symmetries

- **Grand Antiprism**: the 100 vertices achieved by subtracting 20 vertices of  $H_2 \oplus H_2$  from the 120 vertices of the  $H_4$  root system 600-cell – two separate orbits of  $H_2 \oplus H_2$
- This is a semi-regular polytope with automorphism symmetry  $\text{Aut}(H_2 \oplus H_2)$  of order  $400 = 20^2$
- Think of the  $H_2 \oplus H_2$  as coming from the **doubling procedure?** (Likewise for  $\text{Aut}(A_2 \oplus A_2)$  subgroup)
- **Snub 24-cell**:  $2T$  is a subgroup of  $2I$  so subtracting the 24 corresponding vertices of the 24-cell from the 600-cell, one gets a semiregular polytope with 96 vertices and automorphism group  $2T \times 2T$  of order  $576 = 24^2$ .

## Sub root systems

- The above spinor groups had spinor multiplication as the **group operation**
- But also closed under **twisted conjugation** – corresponds to **closure under reflections** (root system property)
- If we take **twisted conjugation** as the group operation instead, we can have various **subgroups**
- These are the remaining **4D root systems** e.g.  $A_4$  or  $B_4$

# Arnold's Trinities

Arnold's observation that many areas of real mathematics can be **complexified** and **quaternionified** resulting in theories with a similar structure.

- The **fundamental trinity** is thus  $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The **projective spaces**  $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The **spheres**  $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The **Möbius/Hopf bundles**  $(S^1 \rightarrow S^1, S^4 \rightarrow S^2, S^7 \rightarrow S^4)$
- The **Lie Algebras**  $(E_6, E_7, E_8)$
- The symmetries of the **Platonic Solids**  $(A_3, B_3, H_3)$
- The **4D groups**  $(D_4, F_4, H_4)$
- **New connections** via my **Clifford spinor construction** (see McKay correspondence)

# Platonic Trinities

- Arnold's connection between  $(A_3, B_3, H_3)$  and  $(D_4, F_4, H_4)$  is **very convoluted** and involves numerous other trinities at intermediate steps:
- **Decomposition of the projective plane** into Weyl chambers and Springer cones
- The **number of Weyl chambers** in each segment is  
 $24 = 2(1 + 3 + 3 + 5), 48 = 2(1 + 5 + 7 + 11), 120 = 2(1 + 11 + 19 + 29)$
- Notice this miraculously **matches the quasihomogeneous weights**  $((2, 4, 4, 6), (2, 6, 8, 12), (2, 12, 20, 30))$  of the Coxeter groups  $(D_4, F_4, H_4)$
- Believe the Clifford connection is **more direct**

# A unified framework for polyhedral groups

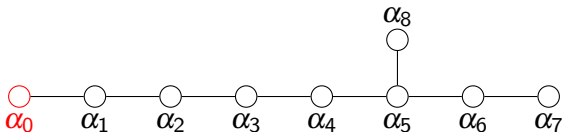
Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm \tilde{A}xA$
$Spin(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$Pin(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the **chiral icosahedral** group has 60 elements, encoded in Clifford by 120 spinors, which form the **binary icosahedral** group
- together with the **inversion/pseudoscalar** / this gives 60 rotations and 60 rotoinversions, i.e. the **full icosahedral** group  $H_3$  in 120 elements (with 240 pinors)
- all three are interesting groups, e.g. in **neutrino and flavour physics** for family symmetry model building

## Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate **conjugacy classes** etc of pinors in Clifford algebra
- Chiral (**binary**) polyhedral groups have irreps
- tetrahedral (12/24):  $1, 1', 1'', 2_s, 2'_s, 2''_s, 3$
- octahedral (24/48):  $1, 1', 2, 2_s, 2'_s, 3, 3', 4_s$
- icosahedral (60/120):  $1, 2_s, 2'_s, 3, \bar{3}, 4, 4_s, 5, 6_s$
- Binary groups are **discrete subgroups of  $SU(2)$**  and all thus have a  $2_s$  spinor irrep
- Connection with the **McKay correspondence!**

# Affine extensions – $E_8^-$



$$-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$$

AKA  $E_8^+$  and along with  $E_8^{++}$  and  $E_8^{+++}$  thought to be the underlying symmetry of **String and M-theory**

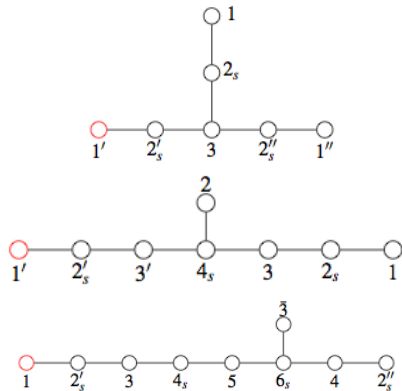
Also interesting from a pure mathematics point of view:  **$E_8$  lattice**, **McKay correspondence** and **Monstrous Moonshine**.

# The McKay Correspondence

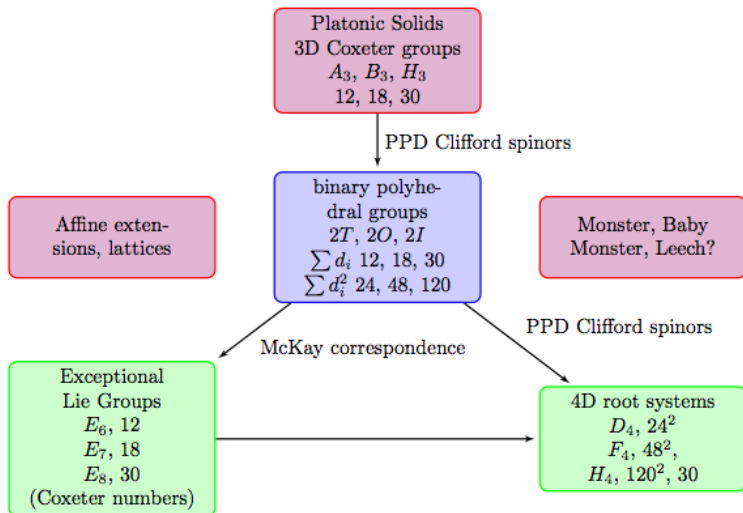
binary polyhedral groups  
 $2T, 2O, 2I$   
 $\sum d_i$  12, 18, 30  
 $\sum d_i^2$  24, 48, 120

McKay correspondence

Exceptional Lie Groups  
 $E_6$ , 12  
 $E_7$ , 18  
 $E_8$ , 30  
 (Coxeter numbers)



# The McKay Correspondence



# The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the **cyclic** and **dicyclic groups** are in correspondence with  $A_n$  and  $D_n$ , e.g. the quaternion group  $Q$  and  $D_4^+$ . So McKay correspondence not just a trinity but **ADE-classification**. We also have  $I_2(n)$  on top of the trinity ( $A_3, B_3, H_3$ )

rank-3 group	diagram	binary	rank-4 group	diagram	Lie algebra	diagram
$A_1 \times A_1 \times A_1$		$Q$	$A_1 \times A_1 \times A_1 \times A_1$		$D_3^+$	
$A_3$		$2T$	$D_4$		$E_6^+$	
$B_3$		$2O$	$F_4$		$E_7^+$	
$H_3$		$2I$	$H_4$		$E_8^+$	

## 4D geometry is surprisingly important for HEP

- 4D root systems are **surprisingly relevant to HEP**
- $A_4$  is  $SU(5)$  and comes up in **Grand Unification**
- $D_4$  is  $SO(8)$  and is the little group of **String theory**
- In particular, its **triality symmetry** is crucial for showing the equivalence of RNS and GS strings
- $B_4$  is  $SO(9)$  and is the little group of **M-Theory**
- $F_4$  is the **largest crystallographic** symmetry in 4D and  $H_4$  is the **largest non-crystallographic** group
- The above are **subgroups** of the latter two
- **Spinorial nature** of the root systems could have **surprising consequences for HEP**

# Quaternions and Clifford Algebra

- The unit **spinors**  $\{1; i, j, k\}$  of  $Cl(3)$  are isomorphic to the **quaternion** algebra  $\mathbb{H}$
- The 3D **Hodge dual of a vector** is a **pure bivector** which corresponds to a **pure quaternion**, and their products are identical (up to sign)

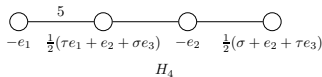
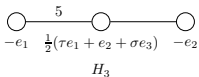
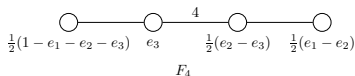
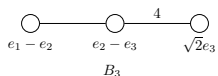
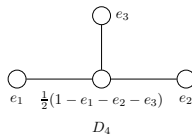
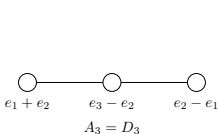
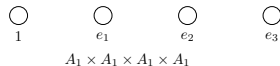
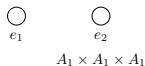
# Discrete Quaternion groups

- The 8 quaternions of the form  $(\pm 1, 0, 0, 0)$  and permutations are called the **Lipschitz units**, and form a realisation of the **quaternion group** in 8 elements.
- The 8 Lipschitz units together with  $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$  are called the **Hurwitz units**, and realise the **binary tetrahedral group** of order 24. Together with the 24 'dual' quaternions of the form  $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$ , they form a group isomorphic to the **binary octahedral group** of order 48.
- The 24 Hurwitz units together with the 96 unit quaternions of the form  $(0, \pm \tau, \pm 1, \pm \sigma)$  and even permutations, are called the **Icosians**. The icosian group is isomorphic to the **binary icosahedral group** with 120 elements.

# Quaternionic representations of 3D and 4D Coxeter groups

- Groups  $E_8$ ,  $D_4$ ,  $F_4$  and  $H_4$  have representations in terms of **quaternions**
- **Extensively used** in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why
- e.g.  $H_4$  consists of 120 elements of the form  $(\pm 1, 0, 0, 0)$ ,  $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$  and  $(0, \pm \tau, \pm 1, \pm \sigma)$
- Seen as remarkable that the **subset of the 30 pure quaternions** is a realisation of  $H_3$  (**a sub-root system**)
- Similarly,  $A_3$ ,  $B_3$ ,  $A_1 \times A_1 \times A_1$  have representations in terms of **pure quaternions**
- Will see there is a **much simpler geometric explanation**

# Quaternionic representations used in the literature



# Demystifying Quaternionic Representations

- 3D: **Pure quaternions** = Hodge dualised (pseudoscalar) **root vectors**
- In fact, they are the **simple roots of the Coxeter groups**
- 4D: **Quaternions** = disguised **spinors** – but those of the **3D Coxeter group** i.e. the binary polyhedral groups!
- This relation between 3D and 4D via the geometric product does not seem to be known
- Quaternion multiplication = ordinary Clifford reflections and rotations

# Demystifying Quaternionic Representations

- **Pure quaternion subset** of 4D groups only gives 3D group if the 3D group **contains the inversion/pseudoscalar  $I$**
- e.g. **does not work** for the tetrahedral group  $A_3$ , but  $A_3 \rightarrow D_4$  **induction still works**, with the central node essentially 'spinorial'
- In fact, it goes the other way around: the **3D groups induce the 4D groups** via spinors
- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as  **$R_1 = \alpha_1 \alpha_2$  and  $R_2 = \alpha_2 \alpha_3$**
- Can see these are '**spinor generators**' and how they don't really contain any more information/roots than the rank-3 groups alone

## Quaternions vs Clifford versors

- **Sandwiching** is often seen as particularly nice feature of the **quaternions giving rotations**
- This is actually a **general feature** of Clifford algebras/versors **in any dimension**; the isomorphism to the **quaternions** is **accidental** to 3D
- However, the **root system** construction does not necessarily generalise
- 2D generalisation merely gives that  $I_2(n)$  is **self-dual**
- **Octonionic** generalisation just induces two copies of the above 4D root systems, e.g.  $A_3 \rightarrow D_4 \oplus D_4$

## References (single-author)

- Clifford algebra unveils a surprising geometric significance of quaternionic root systems of Coxeter groups  
Advances in Applied Clifford Algebras, June 2013, Volume 23, Issue 2, pp 301-321
- A Clifford algebraic framework for Coxeter group theoretic computations (Conference Prize at AGACSE 2012)  
Advances in Applied Clifford Algebras 24 (1). pp. 89-108 (2014)
- Rank-3 root systems induce root systems of rank 4 via a new Clifford spinor construction arXiv:1207.7339 (2012)
- Platonic Solids generate their 4-dimensional analogues  
Acta Cryst. A69 (2013)

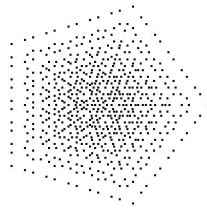
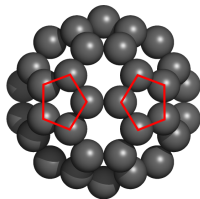
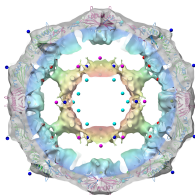
## Conclusions

- Novel **connection** between geometry of **3D and 4D**
- In fact, 3D seems more **fundamental** – contrary to the **usual perspective** of 3D subgroups of 4D groups
- **Spinorial symmetries**
- Clear why **spinor group** gives a root system and why **two factors** of the same group reappear in the **automorphism group**
- Novel **spinorial perspective** on 4D geometry
- **Accidentalness** of the spinor construction and **exceptional** 4D phenomena
- Connection with Arnold's **trinities**, the **McKay correspondence** and **Monstrous Moonshine**

Thank you!

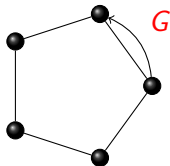
## Motivation: Viruses

- Geometry of **polyhedra** described by **Coxeter** groups
- Viruses have to be '**economical**' with their **genes**
- Encode **structure** modulo **symmetry**
- **Largest discrete symmetry of space** is the **icosahedral** group
- Many other '**maximally symmetric**' objects in nature are also icosahedral: **Fullerenes & Quasicrystals**
- But: viruses are not just polyhedral – they have **radial structure**. **Affine extensions** give **translations**



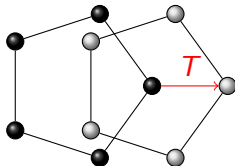
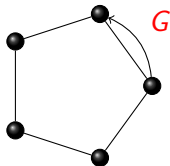
# Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon



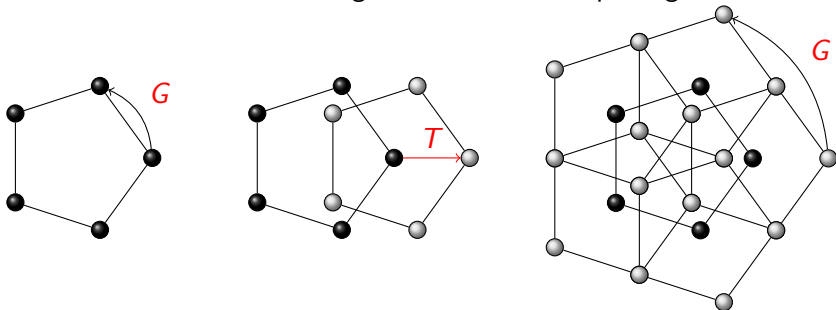
# Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon



# Affine extensions of non-crystallographic root systems

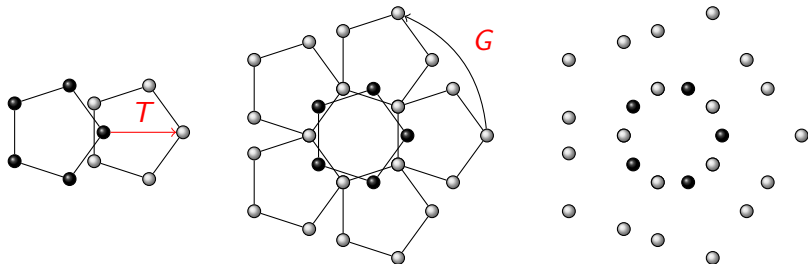
Unit translation along a vertex of a unit pentagon



A **random** translation would give 5 secondary pentagons, i.e. 25 points. Here we have **degeneracies** due to 'coinciding points'.

# Affine extensions of non-crystallographic root systems

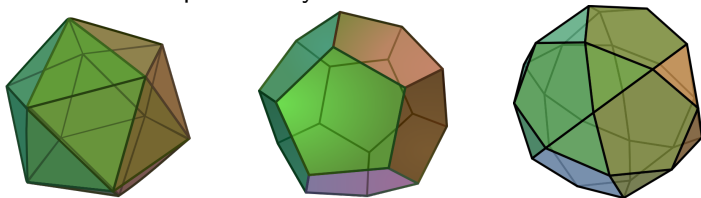
Translation of length  $\tau = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$  (golden ratio)



Looks like a **virus** or **carbon onion**

# Extend icosahedral group with distinguished translations

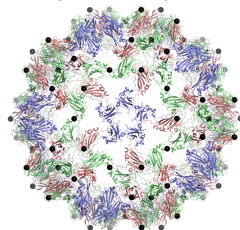
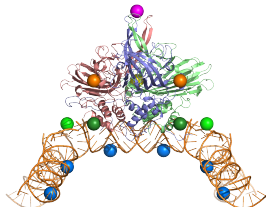
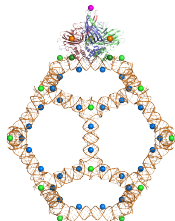
- Radial layers are **simultaneously constrained** by affine symmetry
- Works very well in practice: **finite library of blueprints**
- **Select** blueprint from the **outer shape** (capsid)
- Can **predict inner structure** (nucleic acid distribution) of the virus from the point array



**Affine extensions** of the icosahedral group (giving translations) and their **classification**.

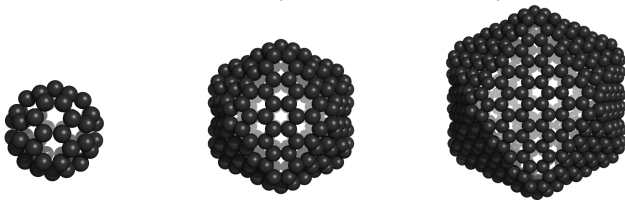
# Use in Mathematical Virology

- Suffice to say **point arrays work very exceedingly well** in practice. Two papers on the mathematical (Coxeter) aspects.
- **Implemented computational problem in Clifford** – some **very interesting mathematics** comes out as well (see later).



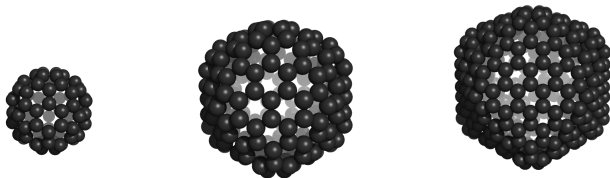
## Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach: **carbon onions** ( $C_{60} - C_{240} - C_{540}$ )



## Extension to fullerenes: carbon onions

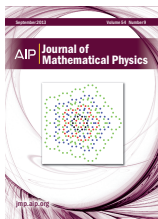
- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach: **carbon onions** ( $C_{80} - C_{180} - C_{320}$ )



## References

- **Novel Kac-Moody-type affine extensions of non-crystallographic Coxeter groups** with Twarock/Bøehm  
J. Phys. A: Math. Theor. 45 285202 (2012)
- **Affine extensions of non-crystallographic Coxeter groups induced by projection** with Twarock/Bøehm  
Journal of Mathematical Physics 54 093508 (2013), [Cover article](#) [September](#)
- **Viruses and Fullerenes – Symmetry as a Common Thread?**  
with Twarock/Wardman/Keef [March Cover](#) Acta Crystallographica A 70 (2). pp. 162-167 (2014), and [Nature Physics Research Highlight](#)

# Applications of affine extensions of non-crystallographic root systems



## Know your onions

Acta Cryst. A 70, 162-167 (2014)

Many viruses have icosahedral symmetry. So do certain carbon onions — Russian doll-like arrangements of nested fullerenes. Pierre-Philippe Dechant and colleagues argue that viruses and carbon onions share the same formation principle: affine symmetry. Imagine a set of points lying on the vertices of a regular pentagon. Duplicate the set, and translate it, then repeatedly rotate the combined set over  $72^\circ$  about the midpoint of the original pentagon. This results in a new set of points obeying five-fold symmetry, yet with a 2D shell structure that is more complex than that of the pentagon: a similar analogue of the (3D) icosahedral group results in a set of points that are nodes in the highly complex protein network structure of, for example, the Parvovirus. Dechant et al. found that affine symmetry explains the structure of experimentally observed carbon onions — a non-trivial result given that all carbon atoms in each of the nested fullerene molecules must be three-connected, that is, bound to three neighbouring carbons. In particular, they identified the extended group that, starting from buckminsterfullerene (the ‘duckyball’), generates the onion  $C_{60}@C_{60}@C_{60}$ . BV

well-known effect for photons, and it turns out to hold for other quantum particles too. James Fickner and colleagues have performed the Hong-Ou-Mandel quantum interference experiment using plasmons, which are quantized surface plasma waves. Pairs of photons are fed into a specially designed photonic waveguide that mixes the paths of the light-excited surface plasmons in the same way as a beam splitter. The outcome is converted back into photons and measured by two detectors. As in the purely photonic case, the characteristic dip in coincidence rate is there, showing that the photons remain indistinguishable when they are converted into plasmons and interfere. (E)

Written by May Chiu, Min Gong, Abigail Oliver, Bart Verbruggen and Alison Wright

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There are interesting applications to **quasicrystals**, **viruses** or **carbon onions**, but here concentrate on the **mathematical** aspects