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Dechant, Pierre-Philippe (2016) A conformal geometric algebra construction of the modular group. In: Alterman Conference on Geometric Algebra, 1st - 9th August 2016, Brasov, Romania. (Unpublished)

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A conformal geometric algebra construction of the modular group

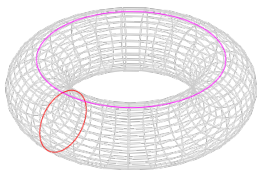
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Alterman Conference Brasov – August 6th, 2016

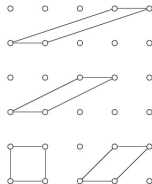
- 1 The modular group and the braid group
- 2 Motivation: Moonshine
- 3 Clifford algebra and the conformal construction
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Torus



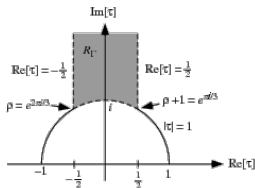
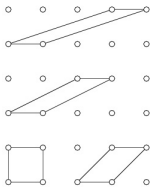
- A **nice manifold**, compact, 2 real-dimensional, or 1-complex dimensional
- The **unique Calabi-Yau 1-fold**
- The product of **two circles**: $T^2 = S^1 \times S^1$
- A torus is actually **topologically flat**, can cut open along the two circles to get a **parallelogram**

Torus embedding in the plane



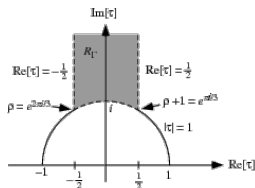
- One can **tile** the plane with parallelograms in many ways (**lattice**)
- One can embed the **torus** if one picks such a lattice and **periodically identifies** opposite sides
- Considering the bottom left hand corner as the origin, and the top right hand corner as a number in the complex plane then this number is called the **complex structure of the torus τ**

Symmetries of the embedding



- Many possible embeddings from the same torus: **redundancy**
- **Winding** around the torus more often $\tau \rightarrow \tau + 1$
- **Flipping** the long and the short side of the torus $\tau \rightarrow -\frac{1}{\tau}$

The modular group



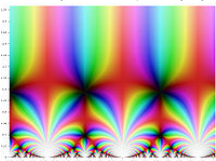
- The **modular generators** $T : \tau \rightarrow \tau + 1$, $S : \tau \rightarrow -\frac{1}{\tau}$ generate the modular group
- They satisfy the **abstract relations** $\langle S, T \mid S^2 = I, (ST)^3 = I \rangle$
($T^\infty = I$)
- Keyhole **fundamental region**

The modular group as $SL(2, \mathbb{Z})$

- $SL(2, \mathbb{Z})$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$
- Subgroup of $SL(2, \mathbb{R})$, and the group of **Moebius** transformations of the plane, i.e. the 2D **conformal** group $PGL(2, \mathbb{C})$

Modular functions

$\mathbb{H} := \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$, $q = \exp(2i\pi\tau)$; $SL(2, \mathbb{R})$ action on \mathbb{H} :

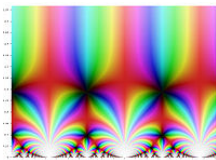


$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1$$

Take G a discrete subgroup of $SL(2, \mathbb{R})$ commensurable with $SL(2, \mathbb{Z})$ then f_G is a **modular function** for G iff

- $f_G : \mathbb{H} \rightarrow \mathbb{C}$ is **meromorphic**
- $f_G(\tau) = f_G\left(\frac{a\tau + b}{c\tau + d}\right) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$
- $f_G(A \cdot \tau) = \sum_{n=-\infty}^{\infty} b_n q^{n/N} \quad \forall A \in SL(2, \mathbb{Z})$ and some N, b_n depending on A with $b_n = 0$ for $n < -M$, $M \in \mathbb{N}$

Modular forms

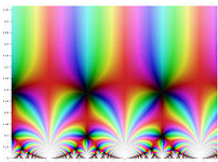


$\Phi(q)$ a **modular form** of $SL(2, \mathbb{Z})$ of **weight** k if

$$\Phi\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \Phi(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

- **Dedekind eta-function** $\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$
- **Theta functions** from an even unimodular lattice L :
 $\theta_L(z) = \sum_{\lambda \in L} \exp \pi i \|\lambda\|^2 z$
- **Eisenstein series** from a lattice Λ : $E_k(\Lambda) = \sum_{0 \neq \lambda \in \Lambda} \lambda^{-k}$ a modular form of weight k

Weak Jacobi forms



weak Jacobi form of weight k and index m

$$\Phi\left(\frac{\mu}{c\mu+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \exp\left(\frac{2i\pi mc\mu^2}{c\tau+d}\right) \Phi(\mu, \tau)$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \mu \in \mathbb{C}, \tau \in \mathbb{H}, z = \exp(2i\pi\mu), q = \exp(2i\pi\tau)$$

$$\Phi(\mu + a\tau + b, \tau) = \exp(-2i\pi m(a^2\tau + 2b\mu)) \Phi(\mu, \tau) \quad \forall a, b \in \mathbb{Z}$$

Modular forms and number theory



- **Number theorists** are very active in modular things
- Connections with **elliptic curves**, Taniyama-Shimura, Andrew Wiles' proof of **Fermat's theorem**
- Extremely clever people, formidably **hard** (a lot worse than what's on the slides)

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Moonshine



- Connection between two very different areas of Mathematics: **finite simple groups** and **modular forms**
- **Moonshine** = crazy, unlikely connection; **insubstantial**; illegal distilling of information from character table
- **Monstrous** = belonging to the Monster group, **enormous**, amazing

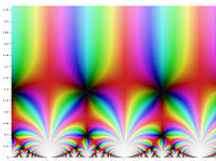
The Monster

- **Finite simple** groups: 18 **series** and 26 **sporadic** (exceptional)
- **Monster**: the **largest** sporadic group
- **Order**:

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \sim 8 \cdot 10^{53}$$
- **194 irreducible representations**: 1, 196883, 21296876, ...
- 20 sporadic groups are subquotients of the Monster: **The Happy Family**
- 6 are not: the **Pariahs**



The Klein j -invariant

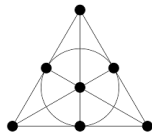


- j -invariant $j(\tau) = \frac{\Theta_{E_8}(\tau)^3}{\eta(\tau)^{24}} - 744$
- **Hauptmodul** for the genus 0 group $G = SL(2, \mathbb{Z})$
- **Ogg**: genus 0 iff p is
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71 offering a bottle
of **Jack Daniel's whiskey**
- Periodic: Fourier expansion **coefficients** wrt $q = e^{2\pi i\tau}$
- $$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots$$

Monstrous Moonshine

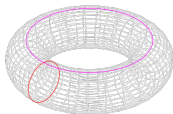
- **Mysterious connection** between two very different areas (**discrete** and **non-discrete**) of Mathematics noticed by **John McKay 1978**
- Monster $1, 196883, 21296876, \dots$
- Klein $j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots$
- $196884 = 196883 + 1, 21493760 = 21296876 + 196883 + 1, \dots$
- Conway, Norton conjectured, Atkin, Fong, and Smith showed a moonshine module exists in 1980 (vertex operator algebra)
- Constructed by Richard **Borcherds** in 1992

The Mathieu group M_{24}



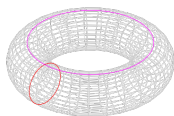
- One of 5 sporadic groups (first to be discovered) named after Mathieu: $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$
- Multiply transitive **permutation** groups
- $244823040 = 3 \cdot 16 \cdot 20 \cdot 21 \cdot 22 \cdot 23 \cdot 24 = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
- Arises as **automorphism group** of the **extended binary Golay code**, **Leech lattice** or **Steiner system** $S(5, 8, 24)$
- 26 irreps 1, 23, 45, 231, 252, 253, 483, 770, 990, 1035, 1265, 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395

K3 – Kähler, Kummer, Kodaira (mountain K2 in Kashmir, André Weil)



- 2 complex-dimensional manifold
- $T^2 \times T^2$ and K3 are the **only Calabi-Yau 2-folds**
- **Kähler** manifold, Ricci-flat, **compact**
- Often used in **string theory compactifications** along with T^2
- From orbifold resolutions, Kummer surfaces etc – all **diffeomorphic**

String Theory



- A supersymmetric **point particle** evolving along a closed **loop** in space-time
- A closed **superstring** sweeps out a 2D **torus**
- **Index of Dirac operator** and its **stringy** generalisation **Ramond operator**
- Compactifications on **tori** and **K3** manifolds
- Interesting (roughly) **modular quantities** and properties arise in string theory: e.g. $N = 4$ **characters**, **elliptic genus** = counts different **states**

K3 elliptic genus

- Known since 80s, **innocuous** enough looking

$$E_{K3}(z, q) = 8 \left(\frac{\theta_2^2(z, q)}{\theta_2^2(1, q)} + \frac{\theta_3^2(z, q)}{\theta_3^2(1, q)} + \frac{\theta_4^2(z, q)}{\theta_4^2(1, q)} \right)$$

- A weak Jacobi form of weight $k = 0$ and index $m = 1$
- Special values give **Euler characteristic**, \hat{A} **Dirac** index, σ **Hirzebruch** signature
- If **rewrite** this in terms of $N = 4$ characters of a specific **string theoretic** non-linear sigma model describing superstring propagation on a **K3** surface (!) one gets

$$E_{K3}(\tau, z) = -2Ch(0; \tau, z) + 20Ch(1/2; \tau, z) + e(q)Ch(\tau, z)$$

- Coefficients in the **q -series** are

$$e(q) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + \dots$$

Mathieu Moonshine – Eguchi, Ooguri, Tachikawa 2010

- **Similar Moonshine phenomenon** connecting finite simple groups and modular forms (ish..)
- **elliptic genus** of an $\mathcal{N} = 4$ SCFT compactified on a **K3-surface** (Taormina, Eguchi 80s)
- Finite simple group: **Mathieu M_{24}** 45,231,770,2277,5796...
- Elliptic genus is

$$E_{K3}(\tau, z) = -2Ch(0; \tau, z) + 20Ch(1/2; \tau, z) + e(q)Ch(\tau, z)$$

- All the coefficients in the q -series

$$e(q) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + \dots$$
 are

twice the dimension of some M_{24} irrep

Mathieu Moonshine – state of the field

- Generally **pretty lost** what's going on
- **Partial results**, e.g. proving coefficients are **positive even integers**, dimensions of M_{24} irreps etc
- **Nothing** in the string theory actually **can have full M_{24} symmetry!**?
- Elliptic genus is invariant under **surfing** across different **Kummer surfaces** despite counting different states in different theories etc e.g. given by symmetry group of D_4 lattice i.e. **binary tetrahedral group**
- Generalisation **umbral moonshine** connecting M_{24} to 23 other symmetry groups of **Niemeyer lattices**

Mathieu Moonshine – state of the field

- **Number theorists** chime in: weakly holomorphic mock modular form of weight $1/2$ on $SL(2, \mathbb{Z})$ with shadow $\eta(q)^3$.
All **extremely technical**
- My construction of **exceptional root systems** from Thursday ties in with these things: lattices, symmetry groups of Kummer surfaces, McKay correspondence, Trinities, Moonshine
- **New approach** to **modular symmetry**?

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Clifford Algebra and orthogonal transformations

- **Inner product** is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting a in b is given by $a' = a - 2(a \cdot b)b = -bab$ (b and $-b$ **doubly cover** the same reflection)
- Via **Cartan-Dieudonné** theorem any orthogonal (/conformal/modular) transformation can be written as **successive reflections**

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1 = \pm A x \tilde{A}$$

- The conformal group $C(p, q) \sim SO(p+1, q+1)$ so can use these for **translations, inversions** etc as well

Conformal Geometric Algebra

- Go to $e_0, e_1, e_2, e_3, e, \bar{e}$, with $e_0^2 = 1, e_i^2 = -1, e^2 = 1, \bar{e}^2 = -1$
- Define two **null** vectors $n \equiv e + \bar{e}, \bar{n} \equiv e - \bar{e}$
- Can **embed** the 4D vector $x = x^\mu e_\mu = te_0 + xe_1 + ye_2 + ze_3$ as a **null vector in 6D** (also normalise $\hat{X} \cdot e = -1$)

$$\hat{X} = \frac{1}{\lambda^2 - x^2} (x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

- So neat thing is that **conformal transformations** are now done by **rotors** (except inversion which is a reflection) – distances are given by **inner products**

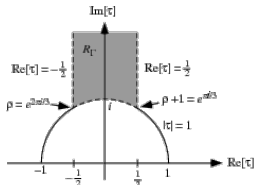
Conformal Transformations in CGA

$$\hat{X} = \frac{1}{\lambda^2 - x^2} (x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

- **Reflection:** spacetime $F(-axa) = -aF(x)a$
- **Rotation:** spacetime $F(Rx\tilde{R}) = RF(x)\tilde{R}$, $R = \exp(\frac{ab}{2\lambda})$
- **Translation:** $F(x+a) = R_T F(x)\tilde{R}_T$ for $R_T = \exp(\frac{na}{2\lambda}) = 1 + \frac{na}{2\lambda}$
- **Dilation:** $F(e^\alpha x) = R_D F(x)\tilde{R}_D$ for $R_D = \exp(\frac{\alpha}{2\lambda} e\bar{e})$
- **Inversion:** Reflection in extra dimension e : $F(\frac{x}{x^2}) = -eF(x)e$
sends $n \leftrightarrow \bar{n}$
- **Special conformal transformation:** $F(\frac{x}{1+ax}) = R_S F(x)\tilde{R}_S$ for
 $R_S = R_I R_T R_I$

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Modular group



- Modular generators: $T : \tau \rightarrow \tau + 1$, $S : \tau \rightarrow -1/\tau$
- $\langle S, T \mid S^2 = I, (ST)^3 = I \rangle$ CGA: $R_Y X \tilde{R}_Y$
- CGA: $T_X = \exp\left(\frac{ne_1}{2}\right) = 1 + \frac{ne_1}{2}$ and $S_X = e_1 e$ (slight issue of complex structure $\tau =$ complex number, not vector in the 2D real plane so map $e_1 : x_1 e_1 + x_2 e_2 \leftrightarrow x_1 + x_2 e_1 e_2 = x_1 + ix_2$)
- $(S_X T_X)^3 = -1$ and $S_X^2 = -1$
- So a 3-fold and a 2-fold rotation in conformal space

Braid group

- $(S_X T_X)^3 = -1$ and $S_X^2 = -1$ is inherently **spinorial**
- Of course Clifford construction gives a **double cover**
- The **braid group** is a double cover
- So **Clifford** construction gives the **braid group double cover** of the **modular group**
- $\sigma_1 = \tilde{T}_X$ and $\sigma_2 = T_X S_X T_X$ satisfying $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$
($= S_X$)

Braid group

- $(S_X T_X)^3 = -1$ and $S_X^2 = -1$ is inherently **spinorial**
- Of course Clifford construction gives a **double cover**
- The **braid group** is a double cover
- So **Clifford** construction gives the **braid group double cover** of the **modular group**
- $\sigma_1 = \tilde{T}_X = \exp(-n\epsilon_1/2)$ and $\sigma_2 = T_X S_X T_X = \exp(-\bar{n}\epsilon_1/2)$ satisfying $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 (= S_X)$
- Might not be known?

Where to go from here?

- Routinely do Clifford **complex analysis** in the plane
- Could look at **meromorphic functions**
- Look at functions in the complex plane in **CGA** representation
- Consider **modular symmetry** in this setup: **modular functions, modular forms, weak Jacobi forms** etc
- Perhaps the spinorial approach opens up **new techniques** to deal with the formidable algebra?