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A systematic construction of representations of quaternionic type

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Alterman Conference Brasov – August 4th, 2016

- 1 Polyhedral groups, Platonic solids and root systems
- 2 A Clifford way of doing orthogonal transformations
- 3 Clifford algebra and quaternions
- 4 Representations from multivector groups: representations of quaternionic type
- 5 Conclusions

Platonic Solids



Platonic Solid	Group	root system
Tetrahedron	A_3 A_1^3	Cuboctahedron Octahedron
Octahedron Cube	B_3	Cuboctahedron + Octahedron
Icosahedron Dodecahedron	H_3	Icosidodecahedron

- **Platonic Solids** have been known for millennia

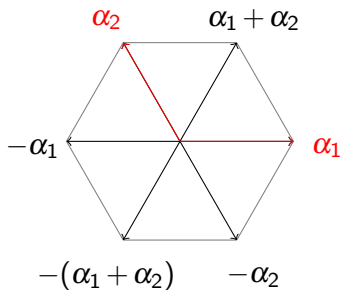
Platonic Solids


 A_1^3
 A_3
 B_3
 H_3

Platonic Solid	Group	root system
Tetrahedron	A_3 A_1^3	Cuboctahedron Octahedron
Octahedron Cube	B_3	Cuboctahedron +Octahedron
Icosahedron Dodecahedron	H_3	Icosidodecahedron

- **Platonic Solids** have been known for millennia
- Described by **Coxeter** groups

Root systems



reflection/Coxeter groups

$$s_{\alpha} : v \rightarrow s_{\alpha}(v) = v - 2 \frac{(v|\alpha)}{(\alpha|\alpha)} \alpha$$

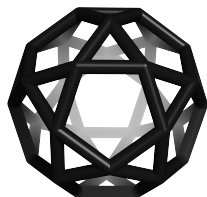
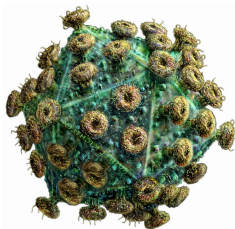
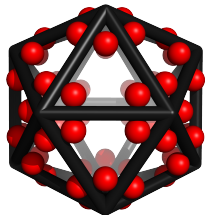
Root system Φ : set of vectors α in a **vector space** with an **inner product** such that

1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$

2. $s_{\alpha}\Phi = \Phi \quad \forall \alpha \in \Phi$

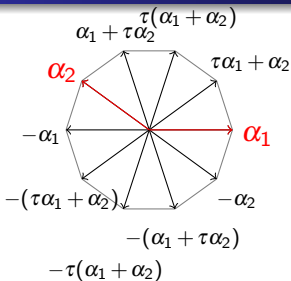
Simple roots: express every element of Φ via a **\mathbb{Z} -linear combination**.

The Icosahedron

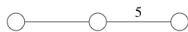


- **Rotational** icosahedral group is $I = A_5$ of order **60**
- **Full** icosahedral group is H_3 of order **120** (including reflections/inversion); generated by the root system icosidodecahedron

Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$



$$A = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate **rotational symmetries of order 5** linear combinations now in the **extended integer ring**

$$\mathbb{Z}[\tau] = \{a + \tau b \mid a, b \in \mathbb{Z}\}$$

golden ratio

$$\tau = \frac{1}{2}(1 + \sqrt{5}) = 2 \cos \frac{\pi}{5}$$

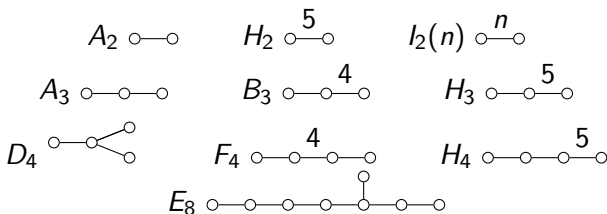
$$x^2 = x + 1$$

$$\tau' = \sigma = \frac{1}{2}(1 - \sqrt{5}) = 2 \cos \frac{2\pi}{5}$$

$$\tau + \sigma = 1, \tau\sigma = -1$$

Cartan-Dynkin diagrams

Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal i.e. angle $\frac{\pi}{2}$, simple link = roots at angle $\frac{\pi}{3}$, link with label $m = \text{angle } \frac{\pi}{m}$.

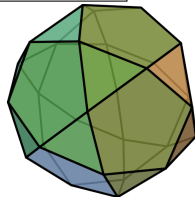
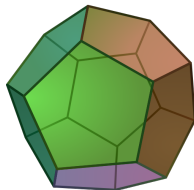
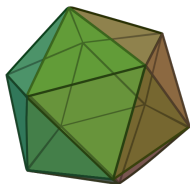


H_3 – the icosahedral group



$$\alpha_1 = (0, 1, 0), \quad \alpha_2 = -\frac{1}{2}(-\sigma, 1, \tau), \quad \alpha_3 = (0, 0, 1)$$

$$T_5 = (\tau, -1, 0), \quad T_3 = (\tau, 0, \sigma), \quad T_2 = (1, 0, 0)$$



Icosahedron, Dodecahedron, Icosidodecahedron (H_3 root system)

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Platonic Solids


 A_1^3
 A_1^4
 A_3
 D_4
 B_3
 F_4
 H_3
 H_4

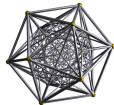
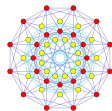
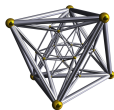
- Concatenating reflections gives **Clifford** spinors (**binary polyhedral groups**)

- These **induce 4D root systems**

$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow$$

$$R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

- 4D analogues of the Platonic Solids and give rise to 4D **Coxeter** groups



Clifford Algebra and orthogonal transformations

- **Geometric Product** for two vectors $ab \equiv a \cdot b + a \wedge b$
- **Inner product** is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting a in n is given by $a' = a - 2(a \cdot n)n = -nan$ (n and $-n$ **doubly cover** the same reflection)
- Via **Cartan-Dieudonné** theorem any orthogonal transformation can be written as **successive reflections**, which are **doubly covered** by Clifford versors/pinors A

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1 =: \pm A x \tilde{A}$$

Clifford Algebra of 3D

- E.g. **Pauli algebra** in 3D (likewise for **Dirac algebra** in 4D) is

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1 e_2 e_3\}}_{1 \text{ trivector}}$$

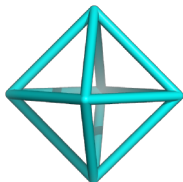
- We can **multiply together root vectors** in this algebra $\alpha_i \alpha_j \dots$
- A general element has **8** components, **even** products (rotations/spinors) have **four** components:

$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow R \tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

- So behaves as a **4D Euclidean** object – inner product

$$(R_1, R_2) = \frac{1}{2}(R_2 \tilde{R}_1 + R_1 \tilde{R}_2)$$

Spinors from reflections



- The 6 **roots** $(\pm 1, 0, 0)$ and permutations in $A_1 \times A_1 \times A_1$ generate 8 **spinors**:
- $\{\pm e_1, \pm e_2, \pm e_3\}$ give the 8 spinors $\{\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1\}$
- This is a **discrete spinor group** isomorphic to the **quaternion** group Q .
- As 4D vectors these are $(\pm 1, 0, 0, 0)$ and permutations, the 8 **roots** of $A_1 \times A_1 \times A_1 \times A_1$ (the 16-cell).

Induction Theorem – root systems

- Induction Theorem: Every rank-3 root system induces a rank-4 root system (and thereby Coxeter groups) via these 3D spinor groups.

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- Check axioms:
 1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$
 2. $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

Induction Theorem – root systems

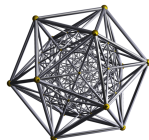
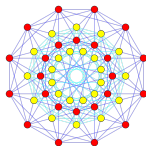
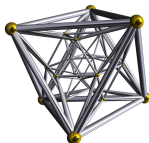
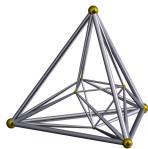
- Induction Theorem: Every rank-3 root system induces a rank-4 root system (and thereby Coxeter groups) via these 3D spinor groups.
- Check axioms:
 1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$
 2. $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$
- Proof: 1. R and $-R$ are in a spinor group by construction (double cover of orthogonal transformations), 2. closure under reflections is guaranteed by the closure property of the spinor group (with a twist: $-R_1 \tilde{R}_2 R_1$)

Spinors from reflections

- Symmetry groups of the **Platonic Solids**:
- The 6/12/18/30 **reflections** in $A_1 \times A_1 \times A_1/A_3/B_3/H_3$ generate 8/24/48/120 **spinors**.
- The **discrete spinor group** is isomorphic to the **quaternion group** Q / **binary tetrahedral group** $2T$ / **binary octahedral group** $2O$ / **binary icosahedral group** $2I$).

Spinors and Polytopes

- Can reinterpret **spinors in \mathbb{R}^3** as **vectors in \mathbb{R}^4**
- Give (exceptional) root systems (D_4, F_4, H_4)
- They constitute the **vertices** of the **16-cell**, **24-cell**, **24-cell** and **dual 24-cell** and the **600-cell**
- These are 4D analogues of the **Platonic Solids**. **Strange symmetries** better understood in terms of **3D spinors**



Root systems in three and four dimensions

The **spinors** from the reflections in the **rank-3 Coxeter group** via the geometric product are the **binary polyhedral groups** Q , $2T$, $2O$ and $2I$, which generate (mostly exceptional) **rank-4 groups**, but **not known why**, and why the 'mysterious symmetries'.

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$	
A_3		$2T$	D_4	
B_3		$2O$	F_4	
H_3		$2I$	H_4	

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Quaternion groups via the geometric product

- The 8 quaternions of the form $(\pm 1, 0, 0, 0)$ and permutations are the **Lipschitz units**, the **quaternion group** in 8 elements.
- The 8 Lipschitz units together with $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ are the **Hurwitz units**, the **binary tetrahedral group** of order 24.
Together with the 24 'dual' quaternions of the form $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$, they form the **binary octahedral group** of order 48.
- The 24 Hurwitz units together with the 96 unit quaternions of the form $(0, \pm \tau, \pm 1, \pm \sigma)$ and even permutations, are called the **Icosians**. The icosian group is isomorphic to the **binary icosahedral group** with 120 elements.
- The unit spinors $\{1; e_2 e_3; e_3 e_1; e_1 e_2\}$ of $\text{Cl}(3)$ are isomorphic to the **quaternion algebra** \mathbb{H} .

H_4 from icosahedral spinors

- The H_3 root system has 30 **roots** e.g. simple roots $\alpha_1 = e_2$, $\alpha_2 = -\frac{1}{2}((\tau - 1)e_1 + e_2 + \tau e_3)$ and $\alpha_3 = e_3$.
- The subgroup of **rotations** is A_5 of order 60
- These are doubly covered by 120 spinors of the form $\alpha_1 \alpha_2 = -\frac{1}{2}(1 - (\tau - 1)e_1 e_2 + \tau e_2 e_3)$, $\alpha_1 \alpha_3 = e_2 e_3$ and $\alpha_2 \alpha_3 = -\frac{1}{2}(\tau - (\tau - 1)e_3 e_1 + e_2 e_3)$.
- As a set of **vectors** in 4D, they are

$(\pm 1, 0, 0, 0)$ (8 permutations), $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ (16 permutations)

$\frac{1}{2}(0, \pm 1, \pm \sigma, \pm \tau)$ (96 even permutations),

which are precisely the 120 roots of the H_4 root system.

Systematic construction of the polyhedral groups

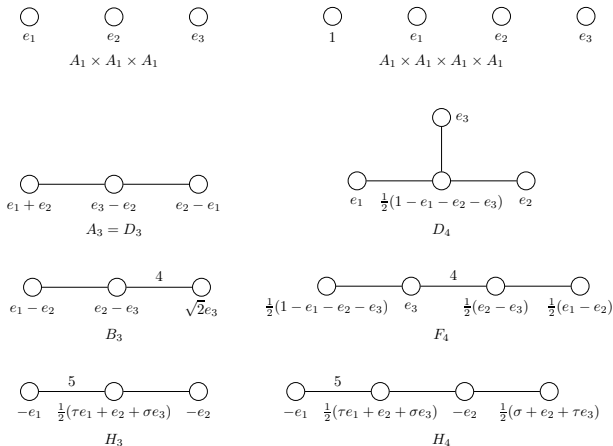
- Multiplying together root vectors in the Clifford algebra gave a systematic way of constructing the binary polyhedral groups as 3D spinors = quaternions.
- The 6/12/18/30 roots in $A_1 \times A_1 \times A_1/A_3/B_3/H_3$ generate 8/24/48/120 spinors.
- The discrete spinor group is isomorphic to the quaternion group Q / binary tetrahedral group $2T$ / binary octahedral group $2O$ / binary icosahedral group $2I$).

A_1^3	A_3	B_3	H_3
A_1^4	D_4	F_4	H_4

Quaternionic representations of 3D and 4D Coxeter groups

- Groups E_8 , D_4 , F_4 and H_4 have representations in terms of **quaternions**
- **Extensively used** in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why
- e.g. H_4 consists of 120 elements of the form $(\pm 1, 0, 0, 0)$, $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and $(0, \pm \tau, \pm 1, \pm \sigma)$
- Seen as remarkable that the **subset of the 30 pure quaternions** is a realisation of H_3 (**a sub-root system**)
- Similarly, B_3 and $A_1 \times A_1 \times A_1$ have representations in terms of **pure quaternions**
- Clifford provides a **much simpler geometric explanation**

Quaternionic representations in the literature



Pure quaternions = Hodge dualised **root vectors**

Quaternions = **spinors**

Demystifying Quaternionic Representations

- **Pure quaternion subset** of 4D groups only gives 3D group if the 3D group **contains the inversion/pseudoscalar I**
- e.g. **does not work** for the tetrahedral group A_3 , but $A_3 \rightarrow D_4$ **induction still works**, with the central node essentially 'spinorial'
- In fact, it goes the other way around: the **3D groups induce the 4D groups** via spinors
- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as **$R_1 = \alpha_1 \alpha_2$ and $R_2 = \alpha_2 \alpha_3$**
- Can see these are '**spinor generators**' and how they don't really contain any more information/roots than the rank-3 groups alone

Quaternions vs Clifford versors

- **Sandwiching** is often seen as particularly nice feature of the **quaternions giving rotations**
- This is actually a **general feature** of Clifford algebras/versors **in any dimension**; the isomorphism to the **quaternions** is **accidental** to 3D
- However, the **root system** construction does not necessarily generalise
- 2D generalisation merely gives that $I_2(n)$ is **self-dual**
- **Octonionic** generalisation just induces two copies of the above 4D root systems, e.g. $A_3 \rightarrow D_4 \oplus D_4$
- Recently constructed E_8 from the **240** pinors doubly covering 120 elements of H_3 in $2^3 = 8$ -dimensional 3D Clifford algebra

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Polyhedral groups as multivector groups

Group	Discrete subgroup	Order	Action Mechanism
$SO(3)$	rotational (chiral)	$ G $	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$2 G $	$x \rightarrow \pm \tilde{A}xA$
$Spin(3)$	binary	$2 G $	$(R_1, R_2) \rightarrow R_1 R_2$
$Pin(3)$	pinory (?)	$4 G $	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the **chiral icosahedral** group has 60 elements, encoded in GA by 120 rotors, which form the **binary icosahedral** group
- together with the **inversion/pseudoscalar** I this gives 60 rotations and 60 rotoinversions, i.e. the **full icosahedral** group H_3 in 120 elements doubly covered by 240 pinors

Some Group Theory: chiral, full, binary, pin

- Easy to calculate **conjugacy classes** etc of versors in GA
- Chiral (**binary**) polyhedral groups have irreps
- tetrahedral (12/24): 1, 1', 1'', 2_s, 2'_s, 2''_s, 3
- octahedral (24/48): 1, 1', 2, 2_s, 2'_s, 3, 3', 4_s
- icosahedral (60/120): 1, 2_s, 2'_s, 3, $\bar{3}$, 4, 4_s, 5, 6_s
- All binary are **discrete subgroups of $SU(2)$** and all thus have a 2_s spinor irrep
- Connection with **Trinities and the McKay correspondence**

Representations from Clifford multivector groups

- The usual picture of **orthogonal transformations** on an n -dimensional vector space is via $n \times n$ **matrices** acting on vectors, immediately making connections with **representations = matrices satisfying the group multiplication laws**.
- **Easy to construct representations** with (s)pinors in the 2^n -dimensional Clifford algebra as **reshuffling components**.
- Spinors leave the **original** n -dimensional **vector** space invariant, **reshuffle** the components of the **vector**.
- But can also consider various representation matrices acting on **different subspaces** of the Clifford algebra.

Representations from Clifford multivector groups – trivial, parity, rotation representations

- The **scalar** subspace is **one-dimensional**. $\tilde{R}1R = \tilde{R}R = 1$ gives the **trivial representation**, and likewise pinors A give the **parity**.
- The double-sided action $\tilde{R}xR$ of spinors R on a **vector** x in the n -dimensional vector space gives an $n \times n$ -dimensional representation, which is just the usual **rotation matrices**.

- E.g. e_1e_2 acting on $x = x_1e_1 + x_2e_2 + x_3e_3$ gives $e_2e_1xe_1e_2 = -x_1e_1 - x_2e_2 + x_3e_3$ which could also be

$$\text{expressed as } \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \\ x_3 \end{pmatrix}$$

- If the spinors were acting as $Rx\tilde{R}$ would give a **potentially different representation**.

Characters, their norm, and the Frobenius-Schur indicator

- **Similarity** transformed representations are also good representations, but are not fundamentally different: they are **equivalent**.
- So want a measure for a representation that is **invariant** under similarity transformations, e.g. the **trace** aka the **character** χ of a matrix
- A **class function** i.e. the same within a conjugacy class because of the cyclicity of the trace
- The **character norm** $\|\chi\|^2 := \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2$
- The **Frobenius-Schur indicator** $\nu := \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$

Real representations of real, complex, and quaternionic type

- $\|\chi\|^2 = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 = 1$: representation of **real** type
- $\|\chi\|^2 = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 = 2$: representation of **complex** type
- $\|\chi\|^2 = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 = 4$: representation of **quaternionic** type
- Theorem: A complex representation is irreducible if and only if $\|\chi\|^2 = 1$.
- Theorem: A **real** representation is **irreducible** if and only if $\|\chi\|^2 + \nu(\chi) = 2$, e.g. $4 - 2 = 2$ or $1 + 1 = 2$.

Representations from Clifford multivector groups – 8×8 and 4×4 (whole algebra / even subalgebra)

- Rather than restricting oneself to the n -dimensional vector space, one can also define representations by $2^n \times 2^n$ -matrices acting on the **whole** Clifford algebra, i.e. any element acting on an arbitrary element, e.g. here 8×8 .
- Likewise, one can define $2^{(n-1)} \times 2^{(n-1)}$ -dimensional spinor representations as acting on the **even subalgebra**.
- 3D spinors have **components** in $(1, e_1 e_2, e_2 e_3, e_3 e_1)$, **multiplication** with another spinor e.g. $e_1 e_2$ will **reshuffle** these components $(e_1 e_2, -1, -e_3 e_1, e_2 e_3)$
- This **reshuffling** can therefore be described by a 4×4 -matrix.

4×4 – explicit example: A_1^3

- E.g. $\boxed{\pm e_1, \pm e_2, \pm e_3}$ give the 8 spinors
- $\boxed{\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1}$, or $(\pm 1, 0, 0, 0)$ (8 permutations)
- $\|\chi\|^2 = 32/8 = 4$, $\nu = -2$ and $\|\chi\|^2 + \nu = 2$ i.e. **real irreducible of quaternionic type**

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Character table of Q

Q	1	-1	$\pm e_1 e_2$	$\pm e_2 e_3$	$\pm e_3 e_1$
1	1	1	1	1	1
1'	1	1	-1	-1	1
1''	1	1	-1	1	-1
1'''	1	1	1	-1	-1
4_H	4	-4	0	0	0

4×4 – explicit example: A_3

- As a set of **vectors** in 4D, they are $(\pm 1, 0, 0, 0)$ (8 permutations), $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ (16 permutations)
- Conjugacy classes:
 $1 \cdot 4^2 + 1 \cdot (-4)^2 + 6 \cdot 0^2 + 8 \cdot 2^2 + 8 \cdot (-2)^2 = 32 + 32 + 32 = 96$
- $\|\chi\|^2 = 96/24 = 4$, $\nu = -2$ and $\|\chi\|^2 + \nu = 2$ i.e. **real irreducible of quaternionic type.**

3×3 – explicit example: H_3

- Icosahedral spinors are

$(\pm 1, 0, 0, 0)$ (8 permutations), $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ (16 permutations)

$\frac{1}{2}(0, \pm 1, \pm \sigma, \pm \tau)$ (96 even permutations),

- E.g. the rotation matrices corresponding to $\alpha_1 \alpha_2$ and $\alpha_2 \alpha_3$ via $\tilde{R}xR$ are

$$\frac{1}{2} \begin{pmatrix} \tau & \tau - 1 & -1 \\ 1 - \tau & -1 & -\tau \\ -1 & \tau & 1 - \tau \end{pmatrix} \text{ and } \frac{1}{2} \begin{pmatrix} \tau & 1 - \tau & -1 \\ 1 - \tau & 1 & -\tau \\ 1 & \tau & \tau - 1 \end{pmatrix}.$$

The characters $\chi(g)$ are obviously 0 and τ

- $\|\chi\|^2 = 120/120 = 1$, $\nu = 1$ and $\|\chi\|^2 + \nu = 2$ i.e. **real irreducible of real type**

3×3 – explicit example: H_3 other way

- If the spinors were acting as $R \times \tilde{R}$, then

$$\frac{1}{2} \begin{pmatrix} \tau & 1-\tau & -1 \\ \tau-1 & -1 & \tau \\ -1 & -\tau & 1-\tau \end{pmatrix} \text{ and } \frac{1}{2} \begin{pmatrix} \tau & 1-\tau & 1 \\ 1-\tau & 1 & \tau \\ -1 & -\tau & \tau-1 \end{pmatrix},$$

with the same characters as before. Swapping the action of the spinor can change the representation.

4×4 – explicit example: H_3

- Spinors $\alpha_1 \alpha_2$ and $\alpha_2 \alpha_3$ multiplying a **generic spinor** $R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2$ from the left **reshuffles** the components (a_1, a_2, a_3, a_0) with the matrices given as

$$\frac{1}{2} \begin{pmatrix} -1 & \tau - 1 & 0 & -\tau \\ 1 - \tau & -1 & -\tau & 0 \\ 0 & \tau & -1 & \tau - 1 \\ \tau & 0 & 1 - \tau & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -\tau & 0 & 1 - \tau & -1 \\ 0 & -\tau & -1 & \tau - 1 \\ \tau - 1 & 1 & -\tau & 0 \\ 1 & 1 - \tau & 0 & -\tau \end{pmatrix},$$

with characters -2 and -2τ .

4×4 – explicit example H_3 : quaternionic type

- 120 4×4 matrices – 9 conjugacy classes, with pairs that have $\pm 2\chi_3$ so gives **4 times** that of the 3×3 case
- $|G| \cdot \|\chi\|^2 = 1 \cdot 4^2 + 1 \cdot (-4)^2 + 12 \cdot (-2\tau)^2 + 12 \cdot (2\tau)^2 + 12 \cdot (-2\sigma)^2 + 12 \cdot (2\sigma)^2 + 20 \cdot (-2)^2 + 20 \cdot (2)^2 + 30 \cdot 0^2 = 480$
- $\|\chi\|^2 = 480/120 = 4$, $v = -2$ and $\|\chi\|^2 + v = 2$ i.e. **real irreducible of quaternionic type**

Character table of $I = A_5$

I	1	$20C_3$	$15C_2$	$12C_5$	$12C_5^2$
1	1	1	1	1	1
3	3	0	-1	τ	σ
$\bar{3}$	3	0	-1	σ	τ
4	4	1	0	-1	-1
5	5	-1	1	0	0

Character table of $2I$

I	1	$20C_3$	$30C_2$	$12C_5$	$12C_5^2$	-1	$-20C_3$	$-12C_5$	$-12C_5^2$
1	1	1	1	1	1	1	1	1	1
3	3	0	-1	τ	σ	3	0	τ	σ
$\bar{3}$	3	0	-1	σ	τ	3	0	σ	τ
4	4	1	0	-1	-1	4	1	-1	-1
5	5	-1	1	0	0	5	-1	0	0
2	2	-1	0	$-\sigma$	$-\tau$	-2	1	σ	τ
2	2	-1	0	$-\tau$	$-\sigma$	-2	1	τ	σ
4	4	1	0	-1	-1	-4	-1	1	1
6	6	0	0	1	1	-6	0	-1	-1
4_H	4	-2	0	-2τ	-2σ	-4	2	2τ	2σ
$4_{\tilde{H}}$	4	-2	0	-2σ	-2τ	-4	2	2σ	2τ

A general construction of representations of quaternionic type – canonical representations

- It had so far been **overlooked** that there is a **systematic construction** of representations of **quaternionic type** for 3D polyhedral groups
- This is simply due to the fact that the **spinors** in 3D provide a realisation of the **quaternions**
- Therefore spinors provide 4x4 representations of quaternionic type for **all** (though limited number of) possible groups
- However, they are **canonical** for a choice of 3D **simple roots**, i.e. there is a preferred amongst all similarity transformed versions
- These **simple roots** also determine the 3x3 **rotation** matrices and their **reversed** representations in a similar **canonical** way

Characters in general

- For a **general spinor** $R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2$ one has **3D character** $\chi = 3a_0^2 - a_1^2 - a_2^2 - a_3^2$ and **representation**

$$\frac{1}{2} \begin{pmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & -2a_0 a_3 + 2a_1 a_2 & 2a_0 a_2 + 2a_1 a_3 \\ 2a_0 a_3 + 2a_1 a_2 & a_0^2 - a_1^2 + a_2^2 - a_3^2 & -2a_0 a_1 + 2a_2 a_3 \\ -2a_0 a_2 + 2a_1 a_3 & 2a_0 a_1 + 2a_2 a_3 & a_0^2 - a_1^2 - a_2^2 + a_3^2 \end{pmatrix}$$

- and the **4D rep and character** are

$$\begin{pmatrix} a_0 & a_3 & -a_2 & a_1 \\ -a_3 & a_0 & a_1 & a_2 \\ a_2 & -a_1 & a_0 & a_3 \\ -a_1 & -a_2 & -a_3 & a_0 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_0 \end{pmatrix} \text{ and } \chi = 4a_0.$$

- Characters** of the representations are **all** determined by the **spinor!**

- 1 Polyhedral groups, Platonic solids and root systems
- 2 A Clifford way of doing orthogonal transformations
- 3 Clifford algebra and quaternions
- 4 Representations from multivector groups: representations of quaternionic type
- 5 Conclusions**

Conclusions

- **General construction** of 4D root systems from 3D root systems – connections with **McKay correspondence**, **Trinities**, **Moonshine** etc
- Construction **systematically** and **canonically** gives representations of 4D root systems and 3D root systems in terms of **(pure) quaternions**
- Construction **systematically** and **canonically** gives construction of the polyhedral groups and their representations, in particular trivial, rotation and spinor representations of **quaternionic type** with relations among them and their characters

Arnold's Trinities

- **Arnold** noticed that often **real**, **complex** and **quaternionic** versions of a theory are remarkably similar
- **Trinities** $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$, $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$, the Möbius/Hopf bundles $(S^1 \rightarrow S^1, S^4 \rightarrow S^2, S^7 \rightarrow S^4)$, (E_6, E_7, E_8)
- **New connection** between (A_3, B_3, H_3) and (D_4, F_4, H_4) (and (E_6, E_7, E_8) !) via my **Clifford spinor construction**
- Also $(24, 48, 120)$, binary polyhedral groups $(2T, 2O, 2I)$ and $(12, 18, 30)$ (see McKay correspondence)

The McKay Correspondence

binary polyhedral groups
 $2T, 2O, 2I$
 $\sum d_i$ 12, 18, 30
 $\sum d_i^2$ 24, 48, 120

McKay correspondence

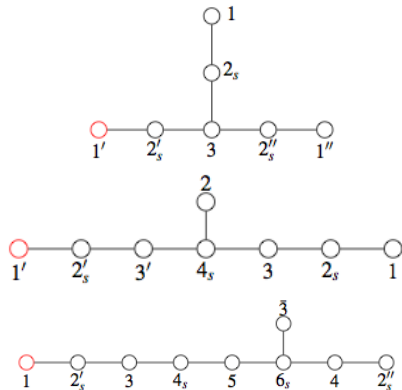
Exceptional
Lie Groups

E_6 , 12

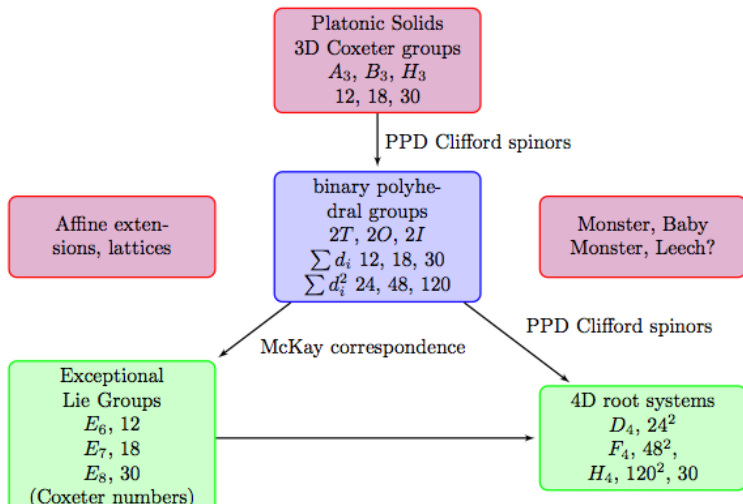
E_7 , 18

E_8 , 30

(Coxeter numbers)



The McKay Correspondence



Some Group Theory: chiral, full, binary, pin

- Easy to calculate **conjugacy classes** etc of versors in GA
- Chiral (**binary**) polyhedral groups have irreps
- tetrahedral (12/24): 1, 1', 1'', 2_s , $2'_s$, $2''_s$, 3
- octahedral (24/48): 1, 1', 2, 2_s , $2'_s$, 3, 3', 4_s
- icosahedral (60/120): 1, 2_s , $2'_s$, 3, $\bar{3}$, 4, 4_s , 5, 6_s
- All binary are **discrete subgroups of $SU(2)$** and all thus have a 2_s spinor irrep
- Connection with **Trinities and the McKay correspondence**

The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the **cyclic** and **dicyclic groups** are in correspondence with A_n and D_n , e.g. the quaternion group Q and D_4^+ . So McKay correspondence not just a trinity but **ADE-classification**. We also have $I_2(n)$ on top of the trinity (A_3, B_3, H_3)

rank-3 group	diagram	binary	rank-4 group	diagram	Lie algebra	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$		D_2^+	
A_3		$2T$	D_4		E_6^+	
B_3		$2O$	F_4		E_7^+	
H_3		$2I$	H_4		E_8^+	