Root systems & Clifford algebras: from symmetries of viruses to E_8 & ADE correspondences

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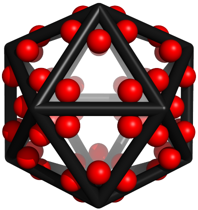
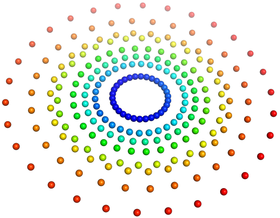
Main results

- New affine symmetry principle for viruses and fullerenes
- H_3 (icosahedral symmetry) induces the E_8 root system
- Each 3D root system induces a 4D root system
- This correspondence extends to various ADE correspondences





Reflection groups: a new approach

- Work at the level of root systems (which define reflection groups)
- Interested in non-crystallographic root systems e.g. viruses, fullerenes etc. But: no Lie algebra, so conventionally less studied
- Clifford algebra is a uniquely suitable framework for reflection groups/root systems: reflection formula, spinor double covers, complex/quaternionic quantities arising as geometric objects

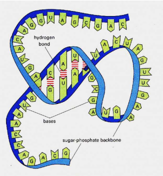
Overview

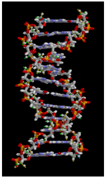
1. Viruses, root systems and affine extensions (with R. Twarock)
   - Viruses
   - Root systems
   - Affine extensions
   - Fullerenes

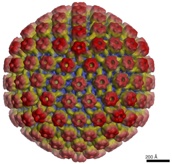
2. Clifford algebras, exceptional root systems and ADE correspondences
   - Clifford basics
   - E_8 from the icosahedron
   - 3D to 4D spinor induction
   - McKay/ADE correspondences

What is a Virus?

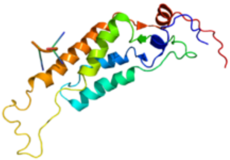
- Transported piece of genetic information that e.g. can run a programme in a host cell
- Genome: RNA or DNA
- Fragile – needs to be protected by a protein shell: capsid
- Gene → mRNA → protein (transcription and translation)
- Each protein = amino acid chain folds into a 3D shape: one geometric building block



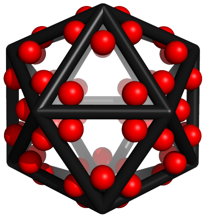


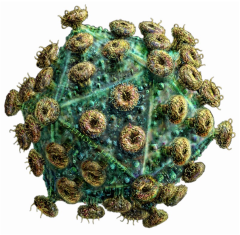


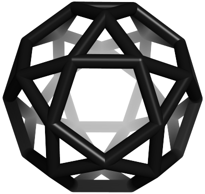
Herpes Simplex Virus type I B-capsids. Data from Rochat et al. J. Virol (2011) 85(4) 1871-4. EMDB #5259 - Rendered with UCSF Chimera. images © 2011 JY Sgro; UW-Madison.



Watson and Crick: The Icosahedron



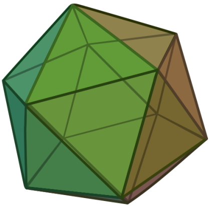


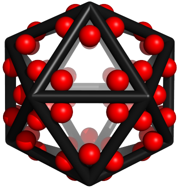


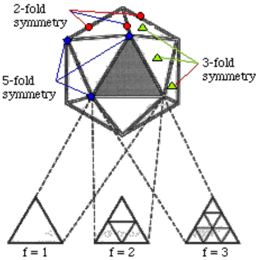
- Crick&Watson: Genetic economy $\rightarrow$ symmetry $\rightarrow$ icosahedral is largest
- Rotational icosahedral group is $I = A_5$ of order 60
- Full icosahedral group is the Coxeter group H_3 of order 120 (including reflections/inversion); generated by the root system icosidodecahedron

Caspar and Klug: Triangulations

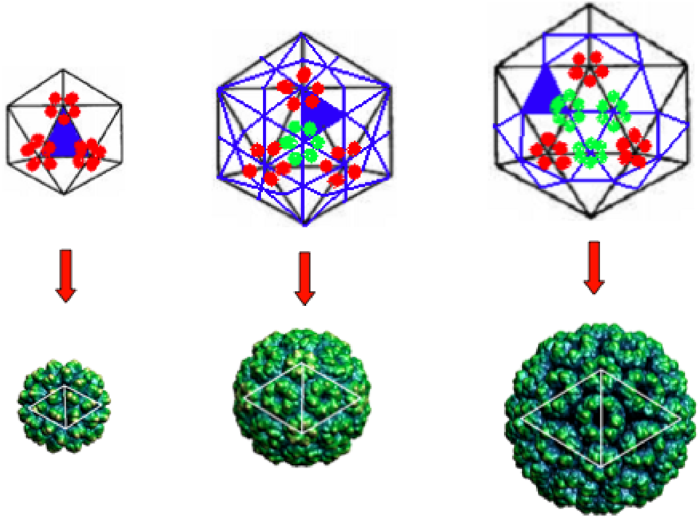
- Mathematical upper limit of 60 for equivalent subunits, but biologically want to do better!
- Gene $\rightarrow$ can already make a triangle $\rightarrow$ might as well make many!
- Caspar-Klug ideas of quasi-equivalence and triangulations



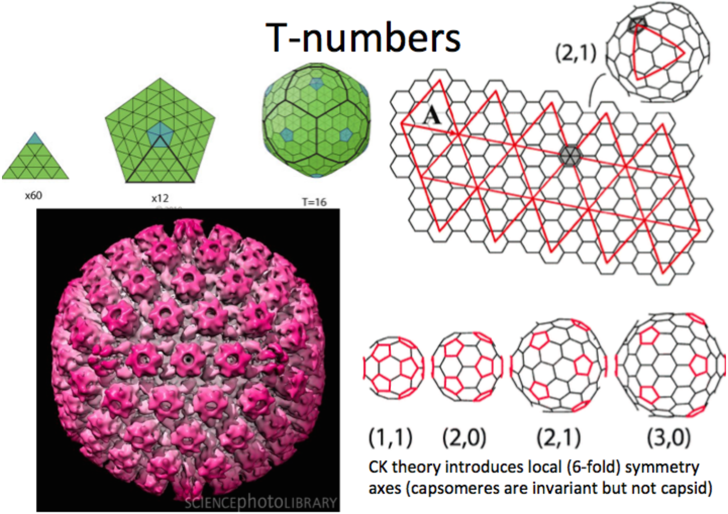




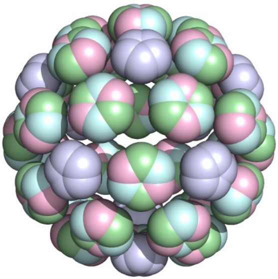
Viruses: Caspar-Klug triangulations

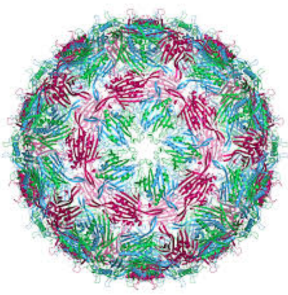


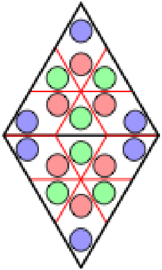
Viruses: Caspar-Klug triangulations



Icosahedral viruses: triangulations and other quasi-equivalent tilings

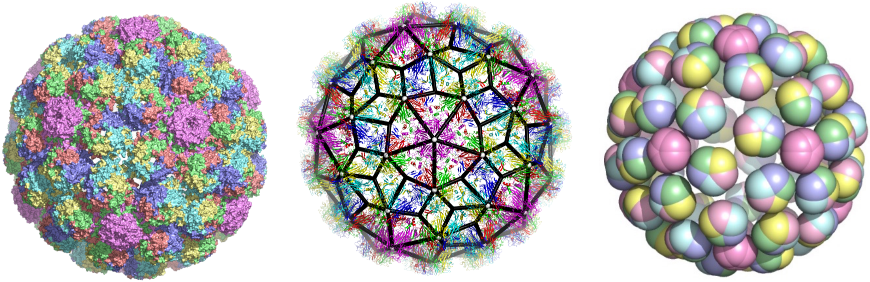






Two viral surface layouts: a $T = 4$ triangulation (e.g. HBV) and a rhombus tiling (MS2) for a pseudo $T = 3$ triangulation

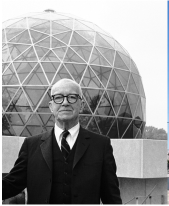
Icosahedral viruses: non-quasiequivalent tilings – Penrose

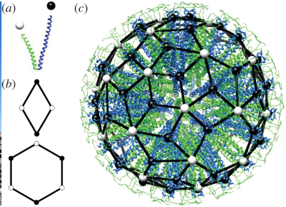


More general icosahedral tilings: Cryo-EM reconstruction of HPV, a kite-rhombus tiling and a pseudo $T = 7$ triangulation.

Other applications

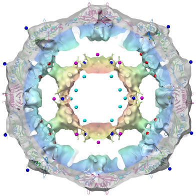
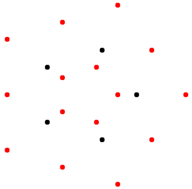
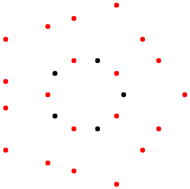
- Architecture: Buckminster Fuller geodesic domes
- The architectural analogue of the kite-rhombus tiling: the new Amazon HQ
- Nanoparticles based on kite-rhombus tiling and local interaction rules



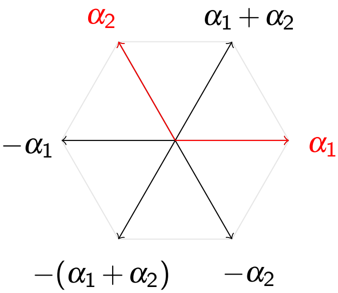


Motivation: Viruses

- Improves the limit to $60T$, but only in terms of surface structures (12 pentagons and rest hexagons).
- Making the symmetry non-compact might allow more general symmetry, simultaneously constraining different 'radial levels'
- Non-compact generator is a translation – motivates looking into affine extensions of icosahedral symmetry
- There is an inherent length scale in the problem – given by size of nucleic acid/protein molecules

Root systems



Root system Φ: set of vectors α in a vector space with an inner product such that

1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall\, \alpha \in \Phi$
2. $s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi$

Simple roots: express every element of Φ via a $\mathbb{Z}$-linear combination.

reflection/Coxeter groups

$$s_\alpha : v \to s_\alpha(v) = v - 2\frac{(v|\alpha)}{(\alpha|\alpha)}\alpha$$

Cartan Matrices

Cartan matrix of α_is is $\boxed{A_{ij} = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} = 2\frac{|\alpha_j|}{|\alpha_i|}\cos\theta_{ij}}$

$$A_2\colon\ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal, simple link = roots at $\frac{\pi}{3}$, link with label m = angle $\frac{\pi}{m}$.

Lie groups to Lie algebras to Coxeter groups to root systems

- Lie group: manifold of continuous symmetries (gauge theories, spacetime)
- Lie algebra: infinitesimal version near the identity
- Non-trivial part is given by a root lattice
- Weyl group is a crystallographic Coxeter group: $\boxed{A_n, B_n/C_n, D_n, G_2, F_4, E_6, E_7, E_8}$ generated by a root system.
- So via this route root systems are always crystallographic. Neglect non-crystallographic root systems $\boxed{I_2(n), H_3, H_4}$.

Affine extensions

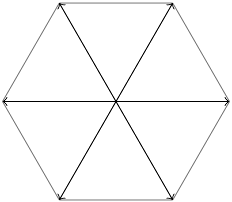
An affine Coxeter group is the extension of a Coxeter group by an affine reflection in a hyperplane not containing the origin $s_{\alpha_0}^{aff}$ whose geometric action is given by

$$s_{\alpha_0}^{aff} v = \alpha_0 + v - \frac{2(\alpha_0|v)}{(\alpha_0|\alpha_0)}\alpha_0$$

Non-distance preserving: includes the translation generator

$$Tv = v + \alpha_0 = s_{\alpha_0}^{aff} s_{\alpha_0} v$$

Affine extensions – A_2



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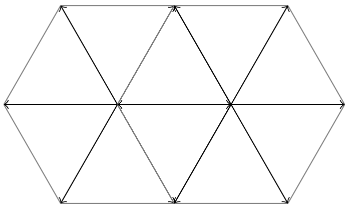
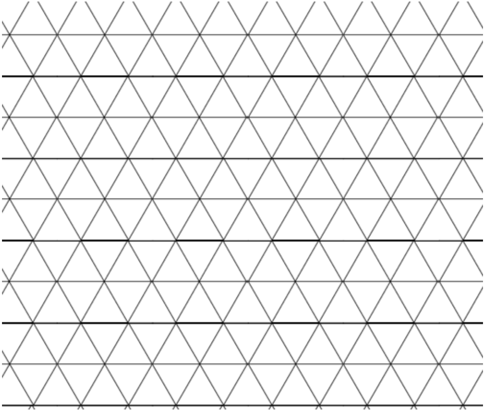
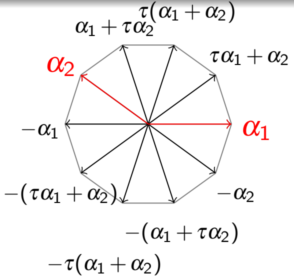


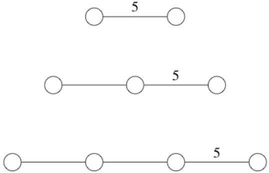
Affine extensions – A_2

Affine extensions of crystallographic Coxeter groups lead to a tessellation of the plane and a lattice.



Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$





$$A=\begin{pmatrix}2 & -\tau\\ -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -\tau\\ 0 & -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -\tau\\ 0 & 0 & -\tau & 2\end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate rotational symmetries of order 5

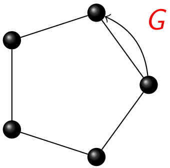
linear combinations now in the extended integer ring

$\mathbb{Z}[\tau]=\{a+\tau b | a,b\in\mathbb{Z}\}$ golden ratio $\tau=\frac{1}{2}(1+\sqrt{5})=2\cos\frac{\pi}{5}$

$x^2=x+1$ $\tau'=\sigma=\frac{1}{2}(1-\sqrt{5})=2\cos\frac{2\pi}{5}$ $\tau+\sigma=1, \tau\sigma=-1$

Affine extensions of non-crystallographic root systems?

Unit translation along a vertex of a unit pentagon



Affine extensions of non-crystallographic root systems?

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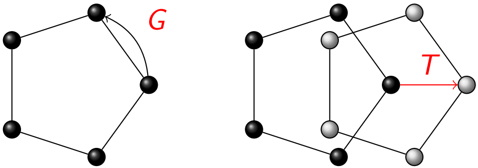

Affine extensions of non-crystallographic root systems?

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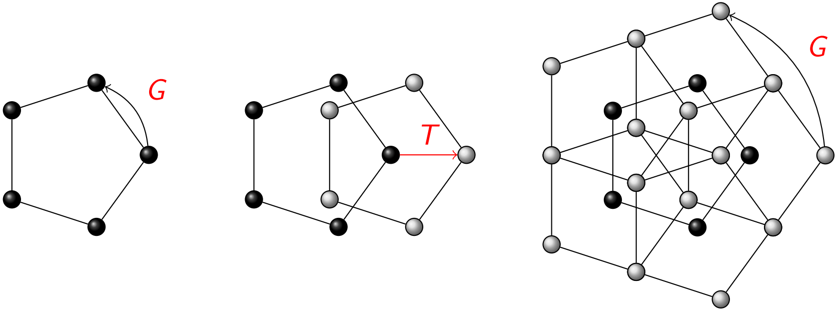
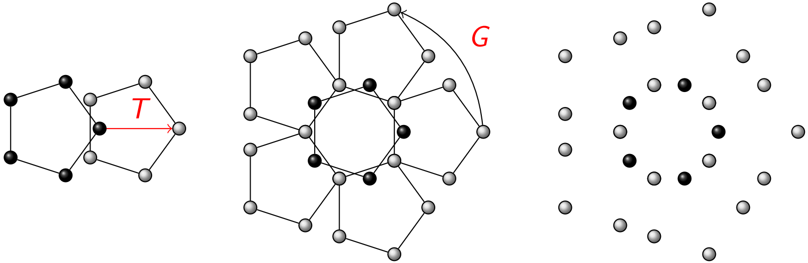


A random translation would give 5 secondary pentagons, i.e. 25 points. Here we have degeneracies due to 'coinciding points'.

Affine extensions of non-crystallographic root systems?

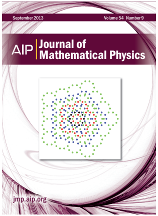
Translation of length $\tau = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$ (golden ratio)

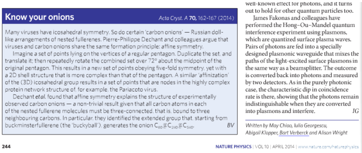


Cartoon version of a virus or carbon onion. Would there be an evolutionary benefit to have more than just compact symmetry? The problem has an intrinsic length scale.

Affine extensions of non-crystallographic Coxeter groups

- 2D and 3D point arrays for applications to viruses, fullerenes, quasicrystals, proteins etc
- Two complementary ways to construct these





Know your onions *Acta Cryst. A* **70,** 162-167 (2014)

Many viruses have icosahedral symmetry. So do certain 'carbon onions' — Russian doll-like arrangements of nested fullerenes. Pierre-Philippe Dechant and colleagues argue that viruses and carbon onions share the same formation principle: affine symmetry.

Imagine a set of points lying on the vertices of a regular pentagon. Duplicate the set, and translate it; then repeatedly rotate the combined set over 72° about the midpoint of the original pentagon. This results in a new set of points obeying five-fold symmetry, yet with a 2D shell structure that is more complex than that of the pentagon. A similar 'affinization' of the (3D) icosahedral group results in a set of points that are nodes in the highly complex protein network structure of, for example, the Pariacoto virus.

Dechant *et al.* found that affine symmetry explains the structure of experimentally observed carbon onions — a non-trivial result given that all carbon atoms in each of the nested fullerene molecules must be three-connected, that is, bound to three neighbouring carbons. In particular, they identified the extended group that, starting from buckminsterfullerene (the 'buckyball'), generates the onion $C_{60}@C_{240}@C_{540}$. *BV*

well-known effect for photons, and it turns out to hold for other quantum particles too.

James Fakonas and colleagues have performed the Hong–Ou–Mandel quantum interference experiment using plasmons, which are quantized surface plasma waves. Pairs of photons are fed into a specially designed plasmonic waveguide that mixes the paths of the light-excited surface plasmons in the same way as a beamsplitter. The outcome is converted back into photons and measured by two detectors. As in the purely photonic case, the characteristic dip in coincidence rate is there, showing that the photons remain indistinguishable when they are converted into plasmons and interfere. *IG*

Written by May Chiao, Iulia Georgescu, Abigail Klopper, Bart Verberck and Alison Wright

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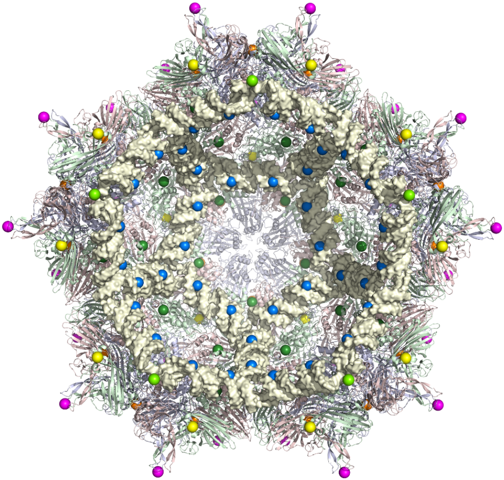
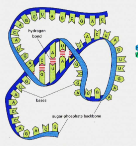
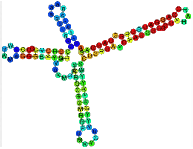
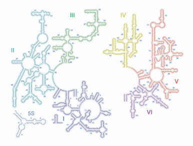
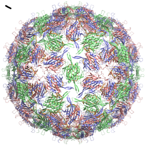
Use in Mathematical Virology



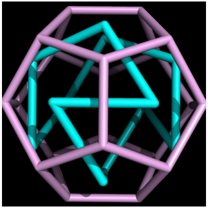
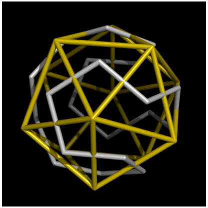
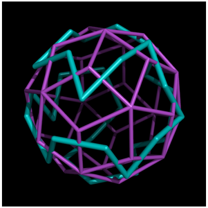
New insight into RNA virus assembly

- There are specific interactions between RNA and coat protein (CP) given by icosahedral symmetry axes
- Essential for assembly, as only this RNA-CP interaction turns CP into right geometric shape for capsid formation
- The RNA forms a Hamiltonian cycle visiting each CP once – dictated by symmetry



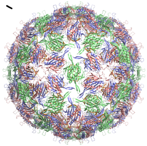
Hamiltonian cycles on icosahedral solids

- So interaction contacts are given by the symmetry
- Orbits of the interaction points have to be visited by the RNA exactly once
- Even the RNA has an icosahedrally ordered component
- Hamiltonian cycles for dodecahedron, icosahedron and rhombic triacontahedron

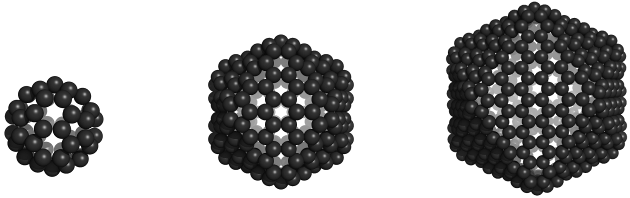
New insight into RNA virus assembly

- More realistic examples for MS has 60 vertices with 41,000 paths
- The RNA is actually circularised by Maturation Protein: only 66 cycles
- With thermodynamical assembly kinetics and 5-fold averaging experiments uniquely idenfied an evolutionarily conserved cycle
- Patents for new antiviral strategies and virus-like nanoparticles e.g. for drug delivery (Twarock group)

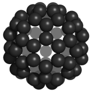
Extension to fullerenes: carbon onions

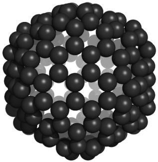
- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{60} - C_{240} - C_{540}$)

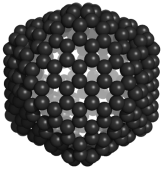


Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{80} - C_{180} - C_{320}$)

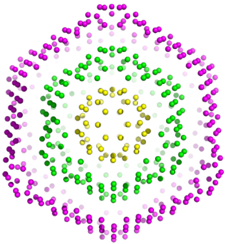


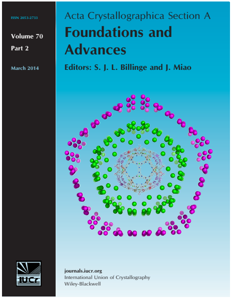


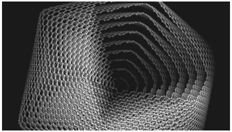


Viruses and fullerenes – symmetry as a common thread?

- Get nested arrangements like Russian dolls: carbon onions (e.g. Nature 510, 250253)

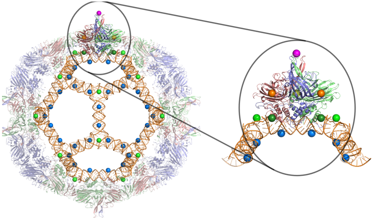


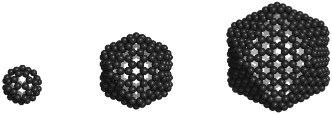




Two major areas for affine extensions of non-crystallographic Coxeter groups

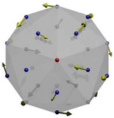
- Non-compact symmetry that relates different structural features in the same polyhedral object when there is an additional length scale
- Novel symmetry principle in Nature, shown that it seems to apply to at least fullerenes and viruses

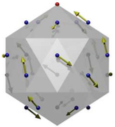


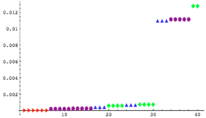


Vibrations of capsids and fullerenes

- Normal modes/vibrations of icosahedral capsids given by representation theory of the icosahedral group
- E.g. $\Gamma^{\text{disp}}_{\text{Icos}} = \Gamma^{1} + 3\Gamma^{3} + \Gamma^{3'} + 2\Gamma^{4} + 3\Gamma^{5}$
- Pioneered by Anne Taormina, Kasper Peeters and Francois Englert







I	1	$20C_3$	$15C_2$	$12C_5$	$12C_5^2$
1	1	1	1	1	1
3	3	0	−1	τ	σ
$\bar{3}$	3	0	−1	σ	τ
4	4	1	0	−1	−1
5	5	−1	1	0	0
Icos perm	12	0	0	2	2
$\chi^{\text{disp}}_{\text{Icos}}$	36	0	0	2τ	2σ
Dodec perm	20	2	0	0	0
$\chi^{\text{disp}}_{\text{Dodec}}$	60	0	0	0	0
IDD perm	30	0	2	0	0
$\chi^{\text{disp}}_{\text{IDD}}$	90	0	−2	0	0
tr Icos perm	60	0	0	0	0
$\chi^{\text{disp}}_{\text{trIcos}}$	180	0	0	0	0

Overview

1. Viruses, root systems and affine extensions (with R. Twarock)
   - Viruses
   - Root systems
   - Affine extensions
   - Fullerenes

2. Clifford algebras, exceptional root systems and ADE correspondences
   - Clifford basics
   - E_8 from the icosahedron
   - 3D to 4D spinor induction
   - McKay/ADE correspondences

Clifford Algebra and orthogonal transformations

- Reflection group setting: vector space with an inner product
- Form an algebra using the product between two vectors

$$\boxed{ab \equiv a \cdot b + a \wedge b}$$

- Inner product is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting x in n is given by $\boxed{x' = x - 2(x \cdot n)n = -nxn}$ (n and $-n$ doubly cover the same reflection)
- Via Cartan-Dieudonné theorem any orthogonal (/conformal/modular) transformation can be written as successive reflections

$$\boxed{x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1} = \pm A x \tilde{A}$$

Clifford Algebra of 3D: geometric objects and algebraic relations

- The Clifford algebra in 3D is

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

- Parallel vectors commute, orthogonal vectors anticommute
- Note e_1e_2 etc square to -1 i.e. are imaginaries representing xy-plane etc i.e. are complex
- $\{1, e_1e_2, e_2e_3, e_3e_1\}$ together satisfy quaternionic relations

Clifford Algebra of 3D: the relation with 4D and 8D

- E.g. Pauli algebra in 3D (likewise for Dirac algebra in 4D) is

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{1 \text{ trivector}}$$

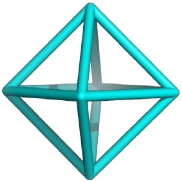
- We can multiply together root vectors in this algebra $\alpha_i\alpha_j\ldots$
- A general element has 8 components (8D vector space), even products (rotations/spinors) have four components (4D subspace):

$$\boxed{R = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2 \Rightarrow R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

- So behaves as a 4D Euclidean object – inner product

$$\boxed{(R_1, R_2) = \frac{1}{2}(R_2\tilde{R}_1 + R_1\tilde{R}_2)}$$

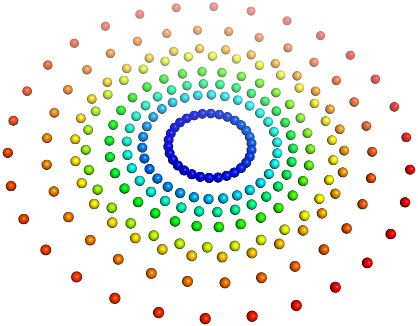
Spinors from reflections



- The 6 roots $(\pm 1, 0, 0)$ and permutations of $A_1 \times A_1 \times A_1$ generate 8 spinors:
- $\boxed{\pm e_1, \pm e_2, \pm e_3}$ give the 8 spinors $\boxed{\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1}$
- This is a discrete spinor group isomorphic to the quaternion group Q.
- As 4D vectors these are $(\pm 1, 0, 0, 0)$ and permutations, the 8 roots of $A_1 \times A_1 \times A_1 \times A_1$ (the 16-cell).

Exceptional E_8 (projected into the Coxeter plane)

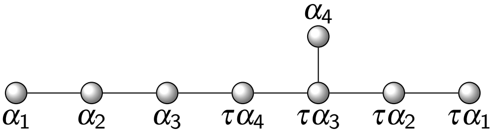
E_8 root system has 240 roots, H_3 has order 120

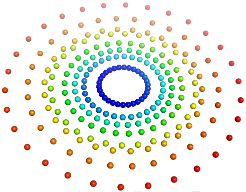


- Order 120 group H_3 doubly covered by 240 (s)pinors in 8D space

- With (somewhat counterintuitive) reduced inner product this gives the E_8 root system

- E_8 is actually hidden within 3D geometry!

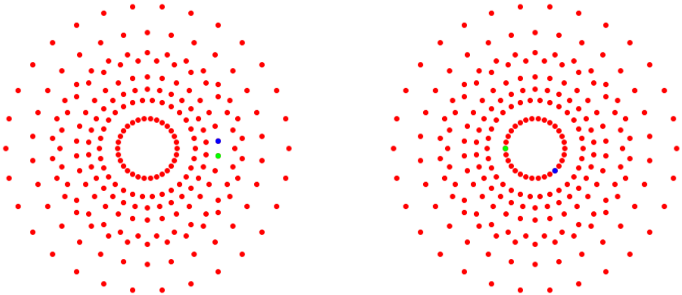




Real Clifford geometry: E_8

- E_8 has exponents $1, 7, 11, 13, 17, 19, 23, 29$, seen as complex eigenvalues of the Coxeter element
- In Clifford algebra, Coxeter element factorises

$$W = \alpha_1 \dots \alpha_8 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{7\pi}{30} B_2\right) \exp\left(\frac{11\pi}{30} B_3\right) \exp\left(\frac{13\pi}{30} B_4\right)$$



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  2. $\boxed{s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi}$

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- In 2D, the space of spinors is also 2D and the root systems are self-dual under an analogous construction

Trinity of 4D Exceptional Root Systems

- Exceptional phenomena: D_4 (triality, important in string theory), F_4 (largest lattice symmetry in 4D), H_4 (largest non-crystallographic symmetry); Exceptional D_4 and F_4 arise from series A_3 and B_3

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$	○ ○ ○	Q	$A_1 \times A_1 \times A_1 \times A_1$	○ ○ ○ ○
A_3	○—○—○	$2T$	D_4	○—○<○○
B_3	○—○—○ (4)	$2O$	F_4	○—○—○—○ (4)
H_3	○—○—○ (5)	$2I$	H_4	○—○—○—○ (5)

Arnold's indirect connection between Trinities (A_3, B_3, H_3) and (D_4, F_4, H_4)

- Arnold had noticed a handwavey connection:
- Decomposition of 3D groups in terms of number of Springer cones matches what are essentially the exponents of the 4D groups:
- A_3: $24 = 2(1+3+3+5)$ – D_4: $(1, 3, 3, 5)$
- B_3: $48 = 2(1+5+7+11)$ – F_4: $(1, 5, 7, 11)$
- H_3: $120 = 2(1+11+19+29)$ – H_4: $(1, 11, 19, 29)$

Arnold's indirect connection between Trinities

rank 4	exponents	W-factorisation
D_4	$1,3,3,5$	$W = \exp\left(\frac{\pi}{6}B_C\right)\exp\left(\frac{\pi}{2}IB_C\right)$
F_4	$1,5,7,11$	$W = \exp\left(\frac{\pi}{12}B_C\right)\exp\left(\frac{5\pi}{12}IB_C\right)$
H_4	$1,11,19,29$	$W = \exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{11\pi}{30}IB_C\right)$

The remaining cases in the root system induction construction work the same way, not just this Trinity! So more general correspondence:

$$(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3) \to (I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$$

Aside for Adriana

rank 4	exponents	W-factorisation
A_4	1, 2, 3, 4	$W = \exp\left(\frac{\pi}{5}B_C\right)\exp\left(\frac{2\pi}{5}IB_C\right)$
B_4	1, 3, 5, 7	$W = \exp\left(\frac{\pi}{8}B_C\right)\exp\left(\frac{3\pi}{8}IB_C\right)$
D_4	1, 3, 3, 5	$W = \exp\left(\frac{\pi}{6}B_C\right)\exp\left(\frac{\pi}{2}IB_C\right)$
F_4	1, 5, 7, 11	$W = \exp\left(\frac{\pi}{12}B_C\right)\exp\left(\frac{5\pi}{12}IB_C\right)$
H_4	1, 11, 19, 29	$W = \exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{11\pi}{30}IB_C\right)$

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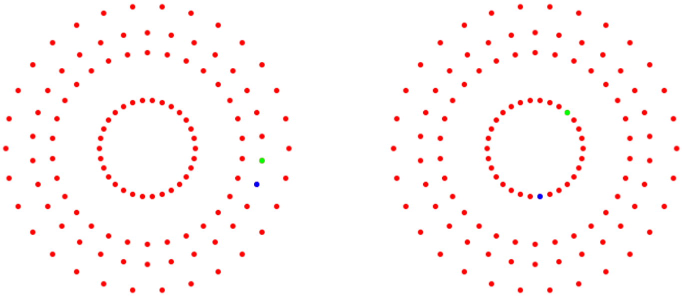
$$(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3) \to (I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$$

4D case: H_4

- E.g. H_4 has exponents $1, 11, 19, 29$
- Coxeter versor decomposes into orthogonal components

$$W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{11\pi}{30}IB_C\right)$$



2D/3D, 2D/4D and ADE correspondences

- McKay correspondence relates even $SU(2)$ subgroups with ADE Lie algebras $(A_{2n-1}, D_{n+2}, E_6, E_7, E_8)$
- Induction theorem: get these as 2D/4D root systems $(I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$ from 2D/3D root systems $(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3)$
- $(2n, 2n+2, 12, 18, 30)$ are numbers of roots, the sum of the dimensions of the irreps and the ADE Coxeter number

4D	G	$\sum d_i$	ADE	h
			$\tilde{A}_{2n-1}$	$2n$
$I_2(n) \times I_2(n)$	Dic_n	$2n+2$	$\tilde{D}_{n+2}$	$2(n+1)$
D_4	$2T$	12	$\tilde{E}_6$	12
F_4	$2O$	18	$\tilde{E}_7$	18
H_4	$2I$	30	$\tilde{E}_8$	30

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2D/3D	$\|\Phi\|$	4D	G	$\sum d_i$	ADE	h
$A_1 \times I_2(n)$	$2n+2$	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$		
A_3	12	D_4	$2T$	12		
B_3	18	F_4	$2O$	18		
H_3	30	H_4	$2I$	30		

2D/3D, 2D/4D and ADE correspondences

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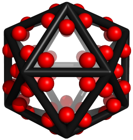
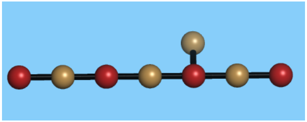
2D/3D	$\|\Phi\|$	4D	G	$\sum d_i$	ADE	h
$I_2(n)$	$2n$	$I_2(n)$	C_{2n}	$2n$	$\tilde{A}_{2n-1}$	$2n$
$A_1 \times I_2(n)$	$2n+2$	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$	$\tilde{D}_{n+2}$	$2(n+1)$
A_3	12	D_4	$2T$	12	$\tilde{E}_6$	12
B_3	18	F_4	$2O$	18	$\tilde{E}_7$	18
H_3	30	H_4	$2I$	30	$\tilde{E}_8$	30

Is there a direct Platonic-ADE correspondence?

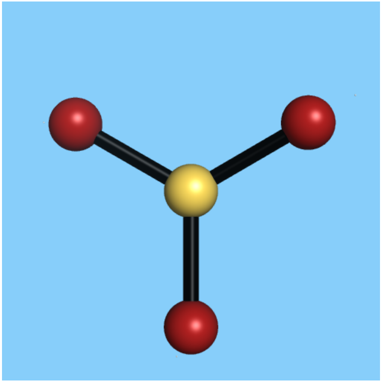




2D/3D		rot	*ADE*		legs
$I_2(n)$	n	n	A_n	$\cdots$	n
$A_1 \times I_2(n)$	n	$2,2,n$	D_{n+2}	$\cdots$	$2,2,n$
A_3		$2,3,3$	E_6		$2,3,3$
B_3	4	$2,3,4$	E_7		$2,3,4$
H_3	5	$2,3,5$	E_8		$2,3,5$

A Trinity of root system ADE correspondences

- 2D/3D root systems $(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3)$
- 2D/4D root systems $(I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$
- ADE root systems $(A_n, D_{n+2}, E_6, E_7, E_8)$



Outlook for algebraic aspects

- All exceptional geometries arise in 3D in a novel Clifford spinorial way, root systems giving rise to Lie groups/algebras
- New view of Coxeter plane geometry: degrees and exponents with geometric interpretation of imaginaries
- A unified framework for doing group and representation theory: polyhedral, orthogonal, conformal, modular (Moonshine) etc
- Conceptual unification at the level of root systems
- ADE correspondences between 2D/3D, 2D/4D & ADE root systems

Thank you!

Clifford Algebra and orthogonal transformations

- Inner product is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting a in b is given by $\boxed{a' = a - 2(a \cdot b)b = -bab}$ (b and $-b$ doubly cover the same reflection)
- Via Cartan-Dieudonné theorem any orthogonal (/conformal/modular) transformation can be written as successive reflections

$$\boxed{x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1} = \pm A x \tilde{A}$$

- The conformal group $C(p,q) \sim SO(p+1, q+1)$ so can use these for translations, inversions etc as well

Conformal Geometric Algebra

- Go to $e_1, e_2, e, \bar{e}$, with $e_0^2 = 1, e_i^2 = -1, e^2 = 1, \bar{e}^2 = -1$
- Define two null vectors $n \equiv e + \bar{e}, \ \bar{n} \equiv e - \bar{e}$
- Can embed the 2D vector $x = x^\mu e_\mu = xe_1 + ye_2$ as a null vector in 4D (also normalise $\hat{X} \cdot e = -1$)

$$\hat{X} = \frac{1}{\lambda^2 - x^2}(x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

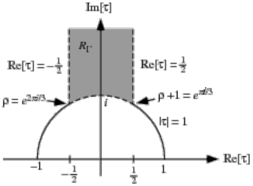
- So neat thing is that conformal transformations are now done by rotors (except inversion which is a reflection) – distances are given by inner products

Conformal Transformations in CGA

$$F(x) = \frac{1}{\lambda^2 - x^2}(x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

- Reflection: spacetime $F(-axa) = -aF(x)a$
- Rotation: spacetime $F(Rx\tilde{R}) = RF(x)\tilde{R}$, $R = \exp(\frac{ab}{2\lambda})$
- Translation: $F(x+a) = R_T F(x)\tilde{R}_T$ for $R_T = \exp(\frac{na}{2\lambda}) = 1 + \frac{na}{2\lambda}$
- Dilation: $F(e^\alpha x) = R_D F(x)\tilde{R}_D$ for $R_D = \exp(\frac{\alpha}{2\lambda}e\bar{e})$
- Inversion: Reflection in extra dimension e: $F(\frac{x}{x^2}) = -eF(x)e$ sends $n \leftrightarrow \bar{n}$
- Special conformal transformation: $F(\frac{x}{1+ax}) = R_S F(x)\tilde{R}_S$ for $R_S = R_I R_T R_I$

Modular group



- Modular generators: $T : \tau \rightarrow \tau + 1$, $S : \tau \rightarrow -1/\tau$
- $\langle S, T | S^2 = I, (ST)^3 = I \rangle$ CGA: $R_Y X \tilde{R}_Y$
- CGA: $T_X = \exp(\frac{ne_1}{2}) = 1 + \frac{ne_1}{2}$ and $S_X = e_1 e$ (slight issue of complex structure τ = complex number, not vector in the 2D real plane so map $e_1 : x_1 e_1 + x_2 e_2 \leftrightarrow x_1 + x_2 e_1 e_2 = x_1 + i x_2$)
- $(S_X T_X)^3 = -1$ and $S_X^2 = -1$
- So a 3-fold and a 2-fold rotation in conformal space

Braid group

- $(S_X T_X)^3 = -1$ and $S_X^2 = -1$ is inherently spinorial
- Of course Clifford construction gives a double cover
- The braid group is a double cover
- So Clifford construction gives the braid group double cover of the modular group
- $\sigma_1 = \tilde{T}_X = \exp(-ne_1/2)$ and $\sigma_2 = T_X S_X T_X = \exp(-\bar{n}e_1/2)$ satisfying $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ $(= S_X)$
- Nice symmetry between the roles of the point at infinity and the origin
- Might not be known? Spinorial techniques might make awkward modular transformations more tractable?