What Clifford algebra can do for Coxeter groups and root systems

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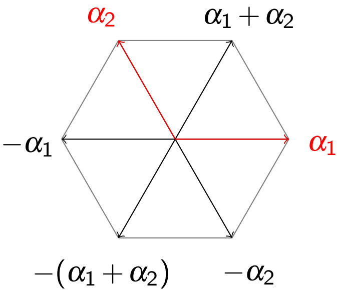
The general theme: Geometry & Symmetry and their Applications

- Worked on a few different things: HEP – strings, particles and cosmology, pure maths and mathematical biology and Clifford algebras and mathematical physics
- Unifying themes of symmetry and geometry (euclidean, conformal, hyperbolic, spherical)
- Continuous Lie groups, e.g. for modeling cosmological spacetimes (Bianchi models)
- Discrete Coxeter groups and Kac-Moody algebras describe gravitational singularities/hidden symmetries in HEP theory, viruses, fullerenes, &c
- Mathematical frameworks of Coxeter groups and Clifford algebras

Overview

Root systems – A_2



Root system Φ: set of vectors α such that

$$\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall\, \alpha \in \Phi}$$

and $\boxed{s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi}$

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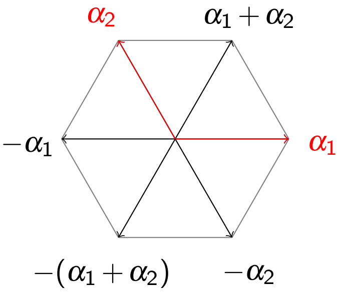

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and $\boxed{s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi}$

Simple roots: express every element of Φ via a $\mathbb{Z}$-linear combination (with coefficients of the same sign).

Cartan Matrices

Cartan matrix of α_is is $\boxed{A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2\frac{|\alpha_j|}{|\alpha_i|}\cos\theta_{ij}}$

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$$\boxed{\cos^2\theta_{ij} = \frac{1}{4}A_{ij}A_{ji}} \quad \boxed{l_j^2 = \frac{A_{ij}}{A_{ji}}l_i^2}$$

$$\boxed{A_{ii} = 2} \quad \boxed{A_{ij} \in \mathbb{Z}^{\leq 0}} \quad \boxed{A_{ij} = 0 \Leftrightarrow A_{ji} = 0}.$$

$$A_2\colon\ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

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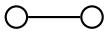
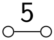
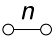
Coxeter-Dynkin diagrams: node $=$ simple root, no link $=$ roots orthogonal, simple link $=$ roots at $\frac{\pi}{3}$, link with label m $=$ angle $\frac{\pi}{m}$.

A_2  H_2 

$I_2(n)$ 

Coxeter groups

A Coxeter group is a group generated by some involutive generators $s_i, s_j \in S$ subject to relations of the form $\boxed{(s_i s_j)^{m_{ij}} = 1}$ with $m_{ij} = m_{ji} \geq 2$ for $i \neq j$.

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The finite Coxeter groups have a geometric representation where the involutions are realised as reflections at hyperplanes through the origin in a Euclidean vector space $\mathscr{E}$. In particular, let $(\cdot|\cdot)$ denote the inner product in $\mathscr{E}$, and v, $\alpha \in \mathscr{E}$.

The generator s_α corresponds to the reflection

$$\boxed{s_\alpha : v \to s_\alpha(v) = v - 2\frac{(v|\alpha)}{(\alpha|\alpha)}\alpha}$$

at a hyperplane perpendicular to the root vector α.

The action of the Coxeter group is to permute these root vectors.

Coxeter groups vs Lie groups vs Lie algebras vs root systems

- Lie group = group and manifold
- Lie algebra = bilinear, antisymmetric bracket and Jacobi identity
- Lie algebras are infinitesimal version of Lie group = near the identity
- Can be more comprehensive e.g. 2D conformal algebra vs 2D conformal group
- But finite group transformation laws can be easier than linearising
- 'Nice' Lie algebras have triangular decomposition: $\mathscr{N}_- \oplus \mathscr{H} \oplus \mathscr{N}_+$

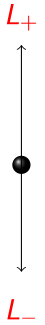
Coxeter groups vs Lie groups vs Lie algebras vs root systems

- 'Nice' Lie algebras have triangular decomposition: $\mathscr{N}_- \oplus \mathscr{H} \oplus \mathscr{N}_+$
- $\mathscr{H}$ is the Cartan subalgebra (maximal commuting = quantum numbers)
- Creation and annihilation algebras $\mathscr{N}$ form root lattice
- Symmetry group is called Weyl group and is a crystallographic Coxeter group: $\boxed{A_n, B_n/C_n, D_n, G_2, F_4, E_6, E_7, E_8}$
- So Coxeter groups in theoretical physics always crystallographic! Neglect $\boxed{I_2(n), H_3, H_4}$.
- Useful Lie algebras are (semi-)simple LA (determinant of Cartan matrix is positive), affine LA (determinant is 0), Kac-Moody algebras, Borcherd's algebras...

Kac-Moody algebras

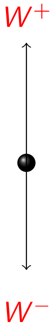
- Kac-Moody algebras $\mathscr{A}$ of rank N are defined by generalised Cartan $(N\times N)$ matrices with $A_{ii}=2$, $A_{ij}\in Z_-(i\neq j)$ and $A_{ij}\neq 0\Rightarrow A_{ji}\neq 0$
- $3N$ generators h_i, e_i, f_i satisfy Chevalley-Serre relations
  $[h_i,h_j]=0$ $[h_i,e_j]=A_{ij}e_j$, $[h_i,f_j]=-A_{ij}f_j$, $[e_i,f_j]=\delta_{ij}h_i$
  $\underbrace{[e_i,[e_i,[\dots,[e_i,e_j]]]\dots]}_{1-A_{ij}\text{ times}}=0$, $\underbrace{[f_i,[f_i,[\dots,[f_i,f_j]]]\dots]}_{1-A_{ij}\text{ times}}=0$
- Simple roots α_i are $[h,e_i]=\alpha_i(h)e_i$

Example – A_1, $SU(2)$, Angular Momentum



- Cartan subalgebra = Quantum number: L_z
- $\mathcal{N}_+$: raising operator $L_+ = \alpha$
- $\mathcal{N}_-$: lowering operator $L_- = -\alpha$
- (L^2 is Casimir/commutes with all algebra elements, is however not actually in the algebra!)

Example – A_1, $SU(2)$, Electroweak



- Cartan subalgebra – Quantum number: A
- $\mathcal{N}_+$: raising operator $W^+ = \alpha$
- $\mathcal{N}_-$: lowering operator $W^- = -\alpha$
- (Since SM electroweak is actually $SU(2) \times U(1)$, $U(1)$ gives another field i, such that physical Z^0 and γ are superpositions of A and i)
- Also $W^\pm$ now charged and self-interact, unlike QED

Affine extensions

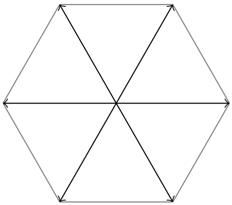
An affine Coxeter group is the extension of a Coxeter group by an affine reflection in a hyperplane not containing the origin $s_{\alpha_0}^{aff}$ whose geometric action is given by

$$\boxed{s_{\alpha_0}^{aff} v = \alpha_0 + v - \frac{2(\alpha_0|v)}{(\alpha_0|\alpha_0)}\alpha_0}$$

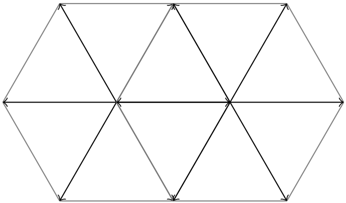
Non-distance preserving: includes the translation generator

$$\boxed{Tv = v + \alpha_0 = s_{\alpha_0}^{aff} s_{\alpha_0} v}$$

Affine extensions – A_2

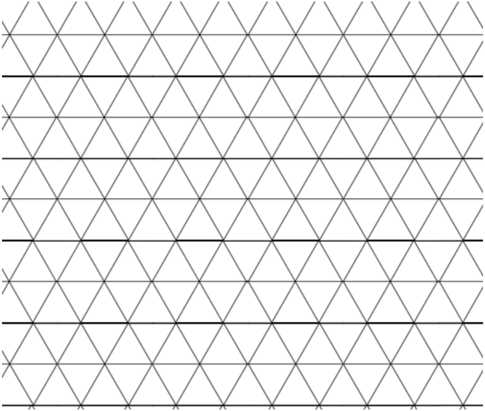


Affine extensions – A_2

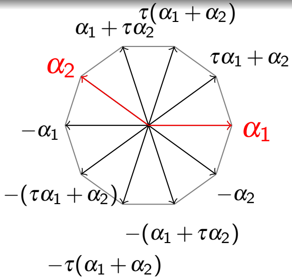


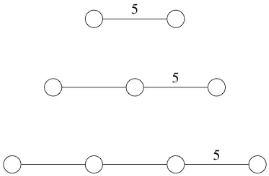
Affine extensions – A_2

Affine extensions of crystallographic Coxeter groups lead to a tessellation of the plane and a lattice.



Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$





$$A=\begin{pmatrix}2 & -\tau\\ -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -\tau\\ 0 & -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -\tau\\ 0 & 0 & -\tau & 2\end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate rotational symmetries of order 5

linear combinations now in the extended integer ring

$\boxed{\mathbb{Z}[\tau]=\{a+\tau b | a,b \in \mathbb{Z}\}}$ golden ratio $\boxed{\tau=\frac{1}{2}(1+\sqrt{5})=2\cos\frac{\pi}{5}}$

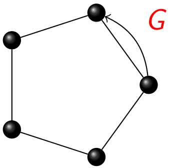
$\boxed{x^2=x+1}$ $\boxed{\tau'=\sigma=\frac{1}{2}(1-\sqrt{5})=2\cos\frac{2\pi}{5}}$ $\boxed{\tau+\sigma=1, \tau\sigma=-1}$

What's new?

- In HEP, mostly come from Lie groups, then Lie algebras, then their Weyl groups and root systems
- This only gives the crystallographic Coxeter groups
- Do the non-crystallographic Coxeter groups have something interesting to offer? In particular, affine extensions?
- Interesting connections between the geometries of different dimensions: Relation between crystallographic and non-crystallographic (E_8 and H_4) and my spinor construction (3 & 4D)
- Both could have interesting consequences for HEP (4D groups and E_8 feature heavily) and other applications (viruses, quasicrystals, proteins, fullerenes...)

Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon



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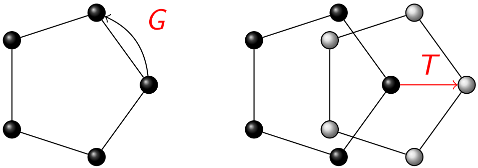

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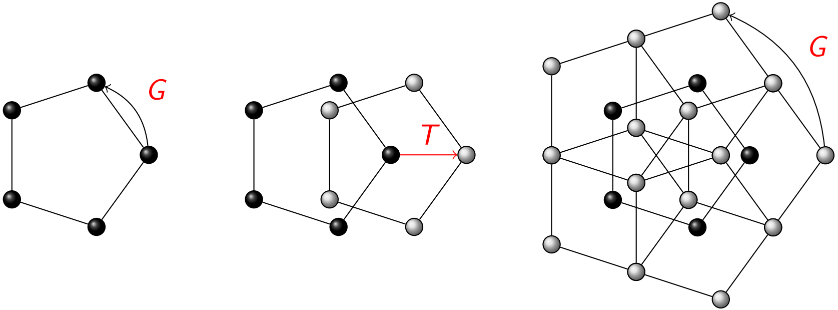
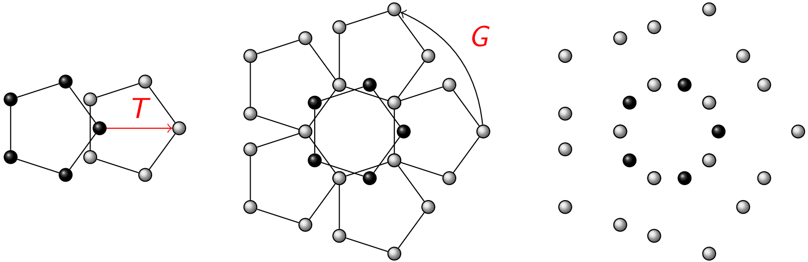


A random translation would give 5 secondary pentagons, i.e. 25 points. Here we have degeneracies due to 'coinciding points'.

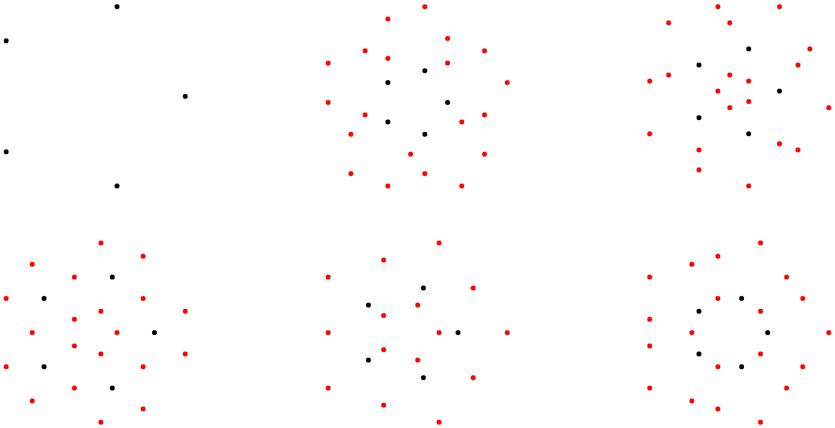
Affine extensions of non-crystallographic root systems

Translation of length $\tau = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$ (golden ratio)



Looks like a virus or carbon onion

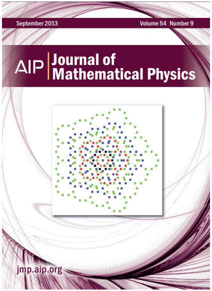
More Blueprints



Road Map

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Crystallographic}\\ \text{Coxeter groups } E_8,\ D_6,\ A_4\end{array}} \arrow[r, "\text{Extend}"] \arrow[d, "\text{Project/Lift}"] & \boxed{\begin{array}{c}\text{Affine extensions}\\ A_4^{=},\ D_6^{>},\ D_6^{=},\ D_6^{<} \text{ and } E_8^{=}\end{array}} \arrow[d, dashed, "\text{Induce (DBT2)}"] \\
\boxed{\begin{array}{c}\text{Non-crystallographic}\\ \text{Coxeter groups } H_4,\ H_3,\ H_2\end{array}} \arrow[r, "\text{Extend (DBT1)}"] & \boxed{\begin{array}{c}\text{Affine extensions}\\ H_2^{=},\ H_3^{>},\ H_3^{=},\ H_3^{<} \text{ and } H_4^{=}\end{array}}
\end{tikzcd}
$$

Applications of affine extensions of non-crystallographic root systems



There are interesting applications to quasicrystals, viruses or carbon onions, but here concentrate on the mathematical aspects

Basics of Clifford Algebra I

- Form an algebra using the Geometric Product
  $\boxed{ab \equiv a \cdot b + a \wedge b}$ for two vectors
- Extend via linearity and associativity to higher grade elements (multivectors)
- For an n-dimensional space generated by n orthogonal unit vectors e_i have 2^n elements
- Then $e_i e_j = e_i \wedge e_j = -e_j e_i$ so anticommute (Grassmann variables, exterior algebra)
- Unlike the inner and outer products separately, this product is invertible
- This feeds through to the differential structure of the theory with more powerful Greens functions methods ∇^{-1}

Basics of Clifford Algebra II

- These are known to have matrix representations over the normed division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ $\Rightarrow$ Classification of Clifford algebras
- E.g. Pauli algebra in 3D (likewise for Dirac algebra in 4D) is

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

- These have the well-known matrix representations in terms of σ- and γ-matrices
- Working with these is not necessarily the most insightful thing to do, so here stress approach to work directly with the algebra
- Naturally have things that square to -1, e.g. $\boxed{(e_1e_2)^2 = e_1e_2e_1e_2 = -e_1e_1e_2e_2 = -1}$, and non-trivial commutation properties

Reflections

- Clifford algebra is very efficient at performing reflections
- Consider reflecting the vector a in a hypersurface with unit normal n:

$$a' = a_\perp - a_\parallel = a - 2a_\parallel = a - 2(a\cdot n)n$$

- c.f. fundamental Weyl reflection $s_i : v \to s_i(v) = v - 2\frac{(v|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i$
- But in Clifford algebra have $n\cdot a = \frac{1}{2}(na + an)$ so reassembles into sandwiching

$$\boxed{a' = -nan}$$

- So both Coxeter and Clifford frameworks are ideally suited to describing reflections – first to combine the two

Reflections and Rotations

- Generate a rotation when compounding two reflections wrt n then m (Cartan-Dieudonné theorem):

$$a'' = mnanm \equiv Ra\tilde{R}$$

where $R = mn$ is called a rotor and a tilde denotes reversal of the order of the constituent vectors ($R\tilde{R} = 1$)

- Now neat thing is all multivectors transform covariantly e.g.

$$MN \to (RM\tilde{R})(RN\tilde{R}) = RM\tilde{R}RN\tilde{R} = R(MN)\tilde{R}$$

so transform double-sidedly

- Rotors form a group, the rotor group, which gives a representation of the Spin group $Spin(n)$ – they transform single-sidedly (obvious now it's a double (universal) cover)

Artin's Theorem and orthogonal transformations

- Artin: every isometry is at most d reflections
- Since have a double cover of reflections (n and $-n$) we have a double cover of $O(p,q)$: $\mathrm{Pin}(p,q)$

$$\boxed{x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1}$$

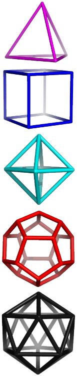
- Pinors/versors = products of vectors $n_1 n_2 \dots n_k$ encode orthogonal transformations via 'sandwiching'
- Cartan-Dieudonné: rotations are an even number of reflections: $\mathrm{Spin}(p,q)$ doubly covers $SO(p,q)$
- The conformal group $C(p,q) \sim SO(p+1,q+1)$ so can use these for translations, inversions etc as well

Spinor techniques

- Of course there is a matrix representation $\underline{R}$ for the action of a spinor: $\boxed{\underline{R}x = Rx\tilde{R}}$
- This is the usual rotation matrix $\underline{R}$ in $SO(p,q)$
- Having the spin double cover/square root of the rotation matrix can be convenient
- E.g. can get differential equations for spinor R that are easier to solve, then can reconstitute $\underline{R}$ if necessary

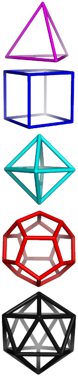
Overview

3D Platonic Solids



- There are 5 Platonic solids
- Tetrahedron (self-dual) (A_3)
- Dual pair octahedron and cube (B_3)
- Dual pair icoshahedron and dodecahedron (H_3)
- Only the octahedron is a root system (actually for (A_1^3))

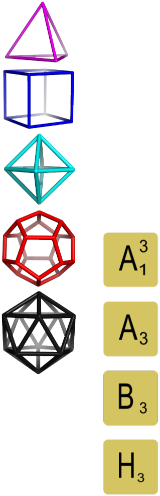
Clifford and Coxeter: Platonic Solids



Platonic Solid	Group	root system
Tetrahedron	A_3 A_1^3	Cuboctahedron Octahedron
Octahedron Cube	B_3	Cuboctahedron +Octahedron
Icosahedron Dodecahedron	H_3	Icosidodecahedron

- Platonic Solids have been known for millennia

Clifford and Coxeter: Platonic Solids



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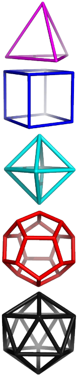
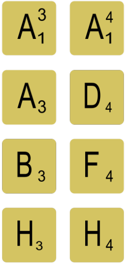
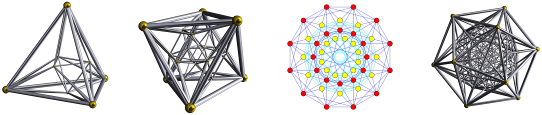
- Described by Coxeter groups

Clifford and Coxeter: Platonic Solids





- Platonic Solids have been known for millennia; described by Coxeter groups
- Concatenating reflections gives Clifford spinors (binary polyhedral groups)
- These induce 4D root systems
  $\psi = a_0 + a_i I e_i \Rightarrow \psi\tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- 4D analogues of the Platonic Solids and give rise to 4D Coxeter groups



4D ‘Platonic Solids’

- In 4D, there are 6 analogues of the Platonic Solids:
- 5-cell (self-dual) (A_4)
- 24-cell (self-dual) (D_4) – a 24-cell and its dual together are the F_4 root system
- Dual pair 16-cell and 8-cell (B_4)
- Dual pair 600-cell and 120-cell (H_4)
- 24-cell, 16-cell and 600-cell are all root systems, as is the related F_4 root system
- 8-cell and 120-cell are dual to a root system, so in 4D out of 6 Platonic Solids only the 5-cell (corresponding to A_n family) is not related to a root system!
- The 4D Platonic solids are not normally thought to be related to the 3D ones except for the boundary cells

Spinorial Symmetries of 4D Polytopes

Spinorial symmetries

rank 3	$\|\Phi\|$	$\|W\|$	rank 4	$\|\Phi\|$	Symmetry
A_3	12	24	D_4 24-cell	24	$2 \cdot 24^2 = 576$
B_3	18	48	F_4 lattice	48	$48^2 = 2304$
H_3	30	120	H_4 600-cell	120	$120^2 = 14400$
A_1^3	6	8	A_1^4 16-cell	8	$3! \cdot 8^2 = 384$
$A_1 \oplus A_2$	8	12	$A_2 \oplus A_2$ prism	12	$12^2 = 144$
$A_1 \oplus H_2$	12	20	$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$A_1 \oplus I_2(n)$	$n+2$	$2n$	$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

Similar for Grand Antiprism (H_4 without $H_2 \oplus H_2$) and Snub 24-cell ($2I$ without $2T$).

Induction Theorem

- Theorem: 3D spinor groups are root systems (R and $-R$ are in a spinor group by construction, and closure under reflections is guaranteed by the closure property of the spinor group)
- Induction Theorem: Every rank-3 root system induces a rank-4 root system.
- Counterexample: not every rank-4 root system is induced in this way
- Spinor group is trivially closed under conjugation, left and right multiplication. Results in non-trivial symmetries when viewed as a polytope/root system.
- Now explains symmetry of the polytopes/root system and thus the order of the rank-4 Coxeter group

Induction Theorem

- So induced 4D polytopes are actually root systems.
- Clear why the number of roots $|\Phi|$ is equal to $|G|$, the order of the spinor group
- Theorem: The automorphism group of the induced root system contains two factors of the respective spinor group acting from the left and the right.
- Only remaining cases in 3D are $A_1 \oplus I_2(n)$, which give $I_2(n) \oplus I_2(n)$

General Case of Induction

Only remaining case is what happens for $A_1 \oplus I_2(n)$ - this gives a doubling $I_2(n) \oplus I_2(n)$

rank 3	rank 4
A_3	D_4
B_3	F_4
H_3	H_4
A_1^3	A_1^4
$A_1 \oplus A_2$	$A_2 \oplus A_2$
$A_1 \oplus H_2$	$H_2 \oplus H_2$
$A_1 \oplus I_2(n)$	$I_2(n) \oplus I_2(n)$

Spinorial Symmetries of 4D Polytopes

Spinorial symmetries

rank 3	$\lvert\Phi\rvert$	$\lvert W\rvert$	rank 4	$\lvert\Phi\rvert$	Symmetry
A_3	12	24	D_4 24-cell	24	$2\cdot 24^2=576$
B_3	18	48	F_4 lattice	48	$48^2=2304$
H_3	30	120	H_4 600-cell	120	$120^2=14400$
A_1^3	6	8	A_1^4 16-cell	8	$3!\cdot 8^2=384$
$A_1\oplus A_2$	8	12	$A_2\oplus A_2$ prism	12	$12^2=144$
$A_1\oplus H_2$	12	20	$H_2\oplus H_2$ prism	20	$20^2=400$
$A_1\oplus I_2(n)$	$n+2$	$2n$	$I_2(n)\oplus I_2(n)$	$2n$	$(2n)^2$

Similar for Grand Antiprism (H_4 without $H_2\oplus H_2$) and Snub 24-cell ($2I$ without $2T$). Additional factors in the automorphism group come from 3D Dynkin diagram symmetries!

Some non-Platonic examples of spinorial symmetries

- Grand Antiprism: the 100 vertices achieved by subtracting 20 vertices of $H_2 \oplus H_2$ from the 120 vertices of the H_4 root system 600-cell – two separate orbits of $H_2 \oplus H_2$
- This is a semi-regular polytope with automorphism symmetry $\mathrm{Aut}(H_2 \oplus H_2)$ of order $400 = 20^2$
- Think of the $H_2 \oplus H_2$ as coming from the doubling procedure? (Likewise for $\mathrm{Aut}(A_2 \oplus A_2)$ subgroup)
- Snub 24-cell: $2T$ is a subgroup of $2I$ so subtracting the 24 corresponding vertices of the 24-cell from the 600-cell, one gets a semiregular polytope with 96 vertices and automorphism group $2T \times 2T$ of order $576 = 24^2$.

Sub root systems

- The above spinor groups had spinor multiplication as the group operation
- But also closed under twisted conjugation – corresponds to closure under reflections (root system property)
- If we take twisted conjugation as the group operation instead, we can have various subgroups
- These are the remaining 4D root systems e.g. A_4 or B_4

What's new?

- Novel connection between geometry of 3D and 4D
- In fact, 3D seems more fundamental – contrary to the usual perspective of 3D subgroups of 4D groups
- Spinorial symmetries
- Clear why spinor group gives a root system and why two factors of the same group reappear in the automorphism group
- Novel spinorial perspective on 4D geometry
- Accidentalness of the spinor construction and exceptional 4D phenomena
- Connection with Arnold's trinities, the McKay correspondence and Monstrous Moonshine

Recap: Clifford algebra and reflections & rotations

- Clifford algebra is very efficient at performing reflections via sandwiching

$$\boxed{a' = -nan}$$

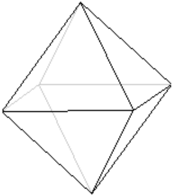
- Generate a rotation when compounding two reflections wrt n then m (Cartan-Dieudonné theorem):

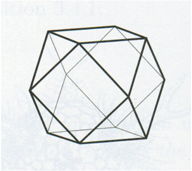
$$\boxed{a'' = mnanm \equiv Ra\tilde{R}}$$

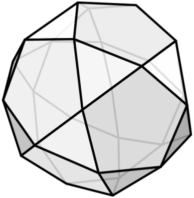
where $R = mn$ is called a rotor and a tilde denotes reversal of the order of the constituent vectors $(R\tilde{R} = 1)$

From the Coxeter simple roots to the root system

- Take the $A_1 \times A_1 \times A_1$ simple roots $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ $\Rightarrow$ under reflections get $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$, the vertices of an octahedron.
- Take the three simple roots of $A_1 \times A_1 \times A_1/A_3/B_3/H_3$. Closure under Clifford reflections generate the whole root system of 6/12/18/30 vertices of an octahedron/cuboctahedron/ cuboctahedron with an octahedron/ icosidodecahedron).







Spinors from reflections

- These are the 3D Coxeter groups that are symmetry groups of the Platonic Solids (tetrahedron and octahedron are similar but simpler than the icosahedron)
- The 6/12/18/30 reflections in $A_1 \times A_1 \times A_1/A_3/B_3/H_3$ generate 8/24/48/120 rotors.
- E.g. $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ give the 8 permutations of $(\pm 1; 0, 0, 0)$ (scalar and bivector parts, the notation will become clear later).
- The discrete spinor group is isomorphic to the quaternion group Q / binary tetrahedral group $2T$/ binary octahedral group $2O$/ binary icosahedral group $2I$).

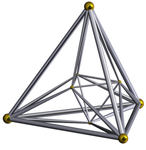
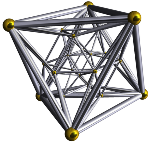
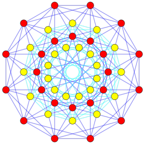
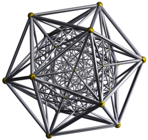
A unified framework for polyhedral groups

Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm\tilde{A}xA$
$\mathrm{Spin}(3)$	binary	$(R_1, R_2) \rightarrow R_1R_2$
$\mathrm{Pin}(3)$	pinor	$(A_1, A_2) \rightarrow A_1A_2$

- e.g. the chiral icosahedral group has 60 elements, encoded in Clifford by 120 rotors, which form the binary icosahedral group
- together with the inversion/pseudoscalar I this gives 60 rotations and 60 rotoinversions, i.e. the full icosahedral group H_3 in 120 elements (with 240 versors)
- all three are interesting groups, e.g. in neutrino and flavour physics for family symmetry model building

Spinors and Polytopes

- The space of $\mathrm{Cl}(3)$-spinors and quaternions have a 4D Euclidean signature: $\psi = a_0 + a_i I e_i \Rightarrow \psi\tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- Can reinterpret spinors in $\mathbb{R}^3$ as vectors in $\mathbb{R}^4$
- Then the spinors constitute the vertices of the 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell
- These are 4D analogues of the Platonic Solids: regular convex 4-polytopes

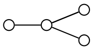





Spinors, Polytopes and Root systems

- The 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell are in fact the root systems of $A_1 \times A_1 \times A_1 \times A_1$, D_4, F_4 and H_4
- Exceptional phenomena: D_4 (triality, important in string theory), F_4 (largest lattice symmetry in 4D), H_4 (largest non-crystallographic symmetry)
- Exceptional D_4 and F_4 arise from series A_3 and B_3
- In fact, can strengthen this statement on inducing polytopes to statement on inducing root systems

Root systems in three and four dimensions

The spinors generated from the reflections contained in the respective rank-3 Coxeter group via the geometric product are realisations of the binary polyhedral groups Q, $2T$, $2O$ and $2I$, which were known to generate (mostly exceptional) rank-4 groups, but not known why, and why the 'mysterious symmetries'.

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$	$\circ \quad \circ \quad \circ$	Q	$A_1 \times A_1 \times A_1 \times A_1$	$\circ \quad \circ \quad \circ \quad \circ$
A_3	$\circ - \circ - \circ$	$2T$	D_4	
B_3	$\circ - \circ \overset{4}{-} \circ$	$2O$	F_4	$\circ - \circ \overset{4}{-} \circ - \circ$
H_3	$\circ - \circ \overset{5}{-} \circ$	$2I$	H_4	$\circ - \circ - \circ \overset{5}{-} \circ$

Arnold's Trinities

Arnold's observation that many areas of real mathematics can be complexified and quaternionified resulting in theories with a similar structure.

- The fundamental trinity is thus $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The projective spaces $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The spheres $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The Möbius/Hopf bundles $(S^1 \to S^1, S^4 \to S^2, S^7 \to S^4)$
- The Lie Algebras (E_6, E_7, E_8)
- The symmetries of the Platonic Solids (A_3, B_3, H_3)
- The 4D groups (D_4, F_4, H_4)
- New connections via my Clifford spinor construction (see McKay correspondence)

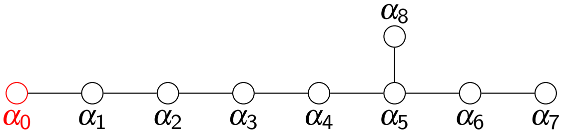
Platonic Trinities

- Arnold's connection between (A_3, B_3, H_3) and (D_4, F_4, H_4) is very convoluted and involves numerous other trinities at intermediate steps:
- Decomposition of the projective plane into Weyl chambers and Springer cones
- The number of Weyl chambers in each segment is $24 = 2(1+3+3+5), 48 = 2(1+5+7+11), 120 = 2(1+11+19+29)$
- Notice this miraculously matches the quasihomogeneous weights $((2,4,4,6),(2,6,8,12),(2,12,20,30))$ of the Coxeter groups (D_4, F_4, H_4)
- Believe the Clifford connection is more direct

Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate conjugacy classes etc of versors in Clifford
- Chiral (binary) polyhedral groups have irreps
- tetrahedral (12/24): $1, 1', 1'', 2_s, 2'_s, 2''_s, 3$
- octahedral (24/48): $1, 1', 2, 2_s, 2'_s, 3, 3', 4_s$
- icosahedral (60/120): $1, 2_s, 2'_s, 3, \bar{3}, 4, 4_s, 5, 6_s$
- All binary are discrete subgroups of $SU(2)$ and all thus have a 2_s spinor irrep
- Connection with the McKay correspondence!

Affine extensions – $E_8^{=}$



$$\boxed{-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8}$$

AKA E_8^+ and along with E_8^{++} and E_8^{+++} thought to be the underlying symmetry of String and M-theory

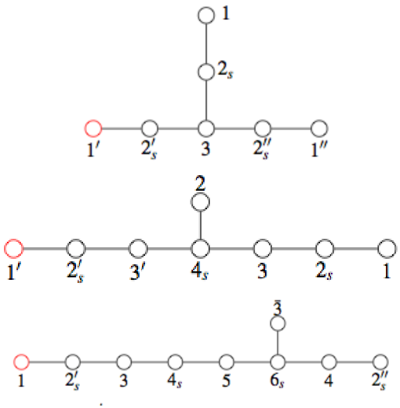
Also interesting from a pure mathematics point of view: E_8 lattice, McKay correspondence and Monstrous Moonshine.

The McKay Correspondence

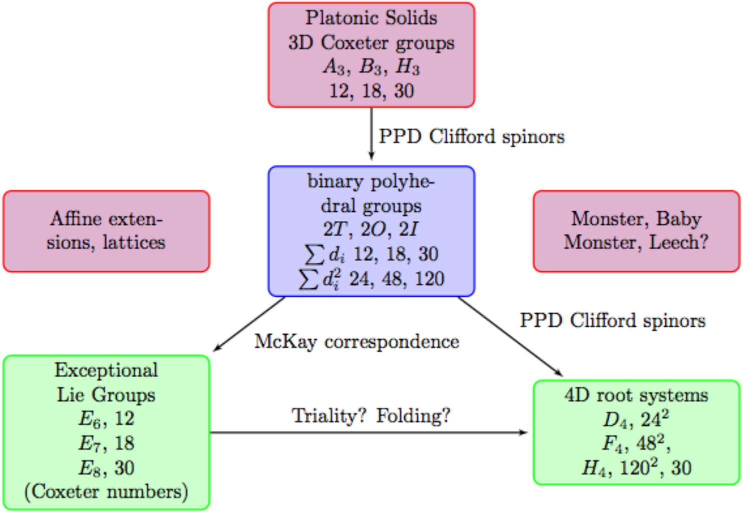
binary polyhedral groups
$2T$, $2O$, $2I$
$\sum d_i$ 12, 18, 30
$\sum d_i^2$ 24, 48, 120

McKay correspondence

Exceptional Lie Groups
E_6, 12
E_7, 18
E_8, 30
(Coxeter numbers)



The McKay Correspondence



The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the cyclic and dicyclic groups are in correspondence with A_n and D_n, e.g. the quaternion group Q and D_4^+. So McKay correspondence not just a trinity but ADE-classification. We also have $I_2(n)$ on top of the trinity (A_3, B_3, H_3)

rank-3 group	diagram	binary	rank-4 group	diagram	Lie algebra	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$		D_4^+	
A_3		$2T$	D_4		E_6^+	
B_3	4	$2O$	F_4	4	E_7^+	
H_3	5	$2I$	H_4	5	E_8^+	

4D geometry is surprisingly important for HEP

- 4D root systems are surprisingly relevant to HEP
- A_4 is $SU(5)$ and comes up in Grand Unification
- D_4 is $SO(8)$ and is the little group of String theory
- In particular, its triality symmetry is crucial for showing the equivalence of RNS and GS strings
- B_4 is $SO(9)$ and is the little group of M-Theory
- F_4 is the largest crystallographic symmetry in 4D and H_4 is the largest non-crystallographic group
- The above are subgroups of the latter two
- Spinorial nature of the root systems could have surprising consequences for HEP

Quaternions and Clifford Algebra

- The unit spinors $\{1; Ie_1; Ie_2; Ie_3\}$ of $\mathrm{Cl}(3)$ are isomorphic to the quaternion algebra $\mathbb{H}$ (up to sign)
- The 3D Hodge dual of a vector is a pure bivector which corresponds to a pure quaternion, and their products are identical (up to sign)

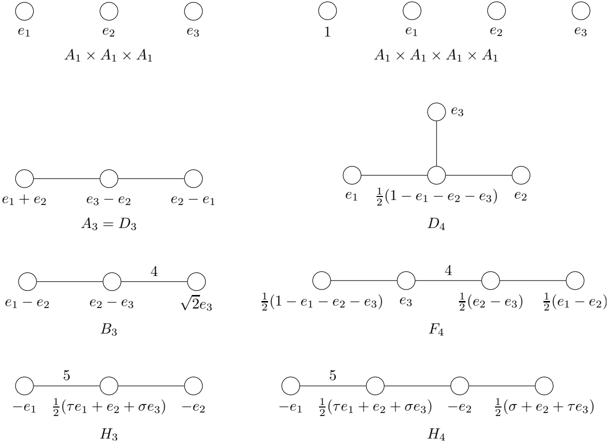
Discrete Quaternion groups

- The 8 quaternions of the form $(\pm 1, 0, 0, 0)$ and permutations are called the Lipschitz units, and form a realisation of the quaternion group in 8 elements.
- The 8 Lipschitz units together with $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ are called the Hurwitz units, and realise the binary tetrahedral group of order 24. Together with the 24 'dual' quaternions of the form $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$, they form a group isomorphic to the binary octahedral group of order 48.
- The 24 Hurwitz units together with the 96 unit quaternions of the form $(0, \pm\tau, \pm 1, \pm\sigma)$ and even permutations, are called the Icosians. The icosian group is isomorphic to the binary icosahedral group with 120 elements.

Quaternionic representations of 3D and 4D Coxeter groups

- Groups E_8, D_4, F_4 and H_4 have representations in terms of quaternions
- Extensively used in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why
- e.g. H_4 consists of 120 elements of the form $(\pm 1, 0, 0, 0)$, $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and $(0, \pm\tau, \pm 1, \pm\sigma)$
- Seen as remarkable that the subset of the 30 pure quaternions is a realisation of H_3 (a sub-root system)
- Similarly, A_3, B_3, $A_1 \times A_1 \times A_1$ have representations in terms of pure quaternions
- Will see there is a much simpler geometric explanation

Quaternionic representations used in the literature



Demystifying Quaternionic Representations

- 3D: Pure quaternions = Hodge dualised (pseudoscalar) root vectors
- In fact, they are the simple roots of the Coxeter groups
- 4D: Quaternions = disguised spinors – but those of the 3D Coxeter group i.e. the binary polyhedral groups!
- This relation between 3D and 4D via the geometric product does not seem to be known
- Quaternion multiplication = ordinary Clifford reflections and rotations

Demystifying Quaternionic Representations

- Pure quaternion subset of 4D groups only gives 3D group if the 3D group contains the inversion/pseudoscalar I
- e.g. does not work for the tetrahedral group A_3, but $A_3 \to D_4$ induction still works, with the central node essentially 'spinorial'
- In fact, it goes the other way around: the 3D groups induce the 4D groups via spinors
- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as $R_1 = \alpha_1\alpha_2$ and $R_2 = \alpha_2\alpha_3$
- Can see these are 'spinor generators' and how they don't really contain any more information/roots than the rank-3 groups alone

Quaternions vs Clifford versors

- Sandwiching is often seen as particularly nice feature of the quaternions giving rotations
- This is actually a general feature of Clifford algebras/versors in any dimension; the isomorphism to the quaternions is accidental to 3D
- However, the root system construction does not necessarily generalise
- 2D generalisation merely gives that $I_2(n)$ is self-dual
- Octonionic generalisation just induces two copies of the above 4D root systems, e.g. $A_3 \rightarrow D_4 \oplus D_4$

References (single-author)

- Clifford algebra unveils a surprising geometric significance of quaternionic root systems of Coxeter groups
  Advances in Applied Clifford Algebras, June 2013, Volume 23, Issue 2, pp 301-321
- A Clifford algebraic framework for Coxeter group theoretic computations (Conference Prize at AGACSE 2012)
  Advances in Applied Clifford Algebras (2013)
- Nomination for W.K. Clifford Prize (2014)
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- Platonic Solids generate their 4-dimensional analogues
  Acta Cryst. A69 (2013)

Coxeter Elements, Degrees and Exponents

- Like the symmetric group, Coxeter groups can have invariant polynomials. Their degrees d are important invariants/group characteristics.
- Turns out that actually degrees d are intimately related to so-called exponents m $\boxed{m = d - 1}$.

Coxeter Elements, Degrees and Exponents

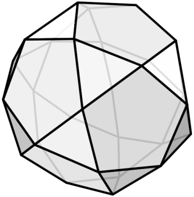
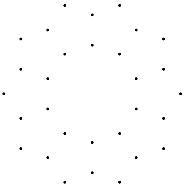
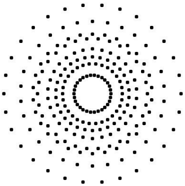
- A Coxeter Element is any combination of all the simple reflections $\boxed{w = s_1 \dots s_n}$, i.e. in Clifford algebra it is encoded by the versor $W = \alpha_1 \dots \alpha_n$ acting as $\boxed{v \rightarrow wv = \pm \tilde{W} v W}$. All such elements are conjugate and thus their order is invariant and called the Coxeter number h.
- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where m are called exponents.
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure (or in fact, there are different ones).

Coxeter Elements, Degrees and Exponents

- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where m are called exponents
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure(s)
- In particular, 1 and $h-1$ are always exponents
- Turns out that actually exponents and degrees are intimately related ($\boxed{m=d-1}$). The construction is slightly roundabout but uniform, and uses the Coxeter plane.

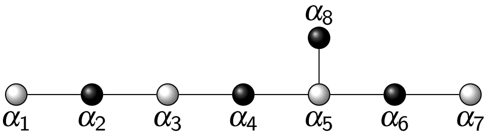
The Coxeter Plane

- Can show every (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by projecting their root system onto the Coxeter plane

The Coxeter Plane

- Obvious from Clifford point of view, that Coxeter element has eigenspaces (eigenblades) rather than just eigenvectors
- In particular, can show every (for our purposes) Coxeter group has a Coxeter plane
- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of mutually commuting generators



The Coxeter Plane

- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of orthogonal = mutually commuting generators but anticommuting root vectors α_w and α_b (duals ω)
- Cartan matrices are positive definite, and thus have a Perron-Frobenius (all positive) eigenvector λ_i.
- Take linear combinations of components of this eigenvector as coefficients of two vectors from the orthogonal sets $v_w = \sum \lambda_w \omega_w$ and $v_b = \sum \lambda_b \omega_b$
- Their outer product/Coxeter plane bivector $\boxed{B_C = v_b \wedge v_w}$ describes an invariant plane where w acts by rotation by $2\pi/h$.

Clifford Algebra and the Coxeter Plane – 2D case

- For $I_2(n)$ take $\boxed{\alpha_1 = e_1,\ \alpha_2 = -\cos\frac{\pi}{n}e_1 + \sin\frac{\pi}{n}e_2}$
- So Coxeter versor is just

$$\boxed{W = \alpha_1\alpha_2 = -\cos\frac{\pi}{n} + \sin\frac{\pi}{n}e_1e_2 = -\cos\frac{\pi}{n} + \sin\frac{\pi}{n}I = -\exp\left(-\frac{\pi I}{n}\right.}$$

- In Clifford algebra it is therefore immediately obvious that the action of the $I_2(n)$ Coxeter element is described by a versor (here a rotor/spinor) that encodes rotations in the e_1e_2-Coxeter-plane and yields $h = n$ since trivially $W^n = (-1)^{n+1}$ yielding $\boxed{w^n = 1}$ via $wv = \tilde{W}vW$.

Clifford Algebra and the Coxeter Plane – 2D case

- So Coxeter versor is just $\boxed{W = -\exp\left(-\frac{\pi I}{n}\right)}$
- $\boxed{I = e_1 e_2}$ anticommutes with both e_1 and e_2 such that sandwiching formula becomes $\boxed{v \rightarrow wv = \tilde{W} v W = \tilde{W}^2 v = \exp\left(\pm\frac{2\pi I}{n}\right) v}$ immediately yielding the standard result for the complex eigenvalues in real Clifford algebra without any need for artificial complexification
- The Coxeter plane bivector $B_C = e_1 e_2 = I$ gives the complex structure
- The Coxeter plane bivector B_C is invariant under the Coxeter versor $\tilde{W} B_C W = \pm B_C$.

Clifford Algebra and the Coxeter Plane – 3D case

- In 3D, A_3, B_3, H_3 have $\{1,2,3\}$, $\{1,3,5\}$ and $\{1,5,9\}$
- Coxeter element is product of a spinor in the Coxeter plane with the same complex structure as before, and a reflection perpendicular to the plane
- So in 3D still completely determined by the plane
- 1 and $h-1$ are rotations in Coxeter plane
- $h/2$ is the reflection (for v in the normal direction)

$$\boxed{wv = \tilde{W}^2 = \exp(\pm\frac{2\pi I}{h}\frac{h}{2}) = \exp(\pm\pi I)v = -v}$$

Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can factorise W into orthogonal (commuting/anticommuting) components $\boxed{W = \alpha_1 \dots \alpha_n = W_1 \dots W_n}$ with $\boxed{W_i = \exp(\pi m_i I_i / h)}$
- Here, I_i is a bivector describing a plane with $\boxed{I_i^2 = -1}$
- For v orthogonal to the plane descrbed by I_i we have $\boxed{v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i W_i v = v}$ so cancels out
- For v in the plane we have $\boxed{v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i / h) v}$
- Thus if we decompose W into orthogonal eigenspaces, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

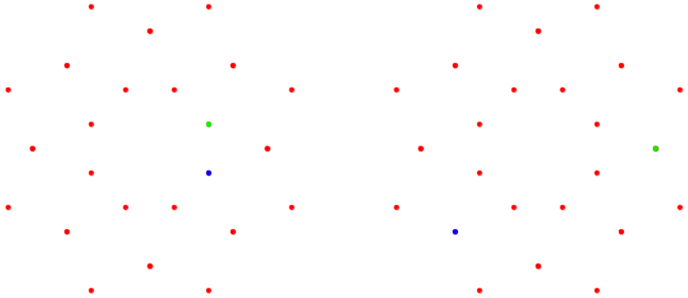
Clifford algebra: no need for complexification

- For v in the plane we have
  $$\boxed{v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i/h)v}$$
- So complex eigenvalue equation arises geometrically without any need for complexification
- Different complex structures immediately give different eigenplanes
- Eigenvalues/angles/exponents given from just factorising $W = \alpha_1 \dots \alpha_n$
- E.g. B_4 has exponents $1, 3, 5, 7$ and $W = \exp\left(\frac{\pi}{8} I_1\right)\exp\left(\frac{3\pi}{8} I_2\right)$
- Here we have been looking for orthogonal eigenspaces, so innocuous – different complex structures commute
- But not in general – naive complexification can be misleading

Clifford Algebra and the Coxeter Plane – 4D case

- E.g. B_4 has exponents $1, 3, 5, 7$
- Coxeter versor decomposes into orthogonal components

$$\boxed{W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{8}B_C\right)\exp\left(\frac{3\pi}{8}IB_C\right)}$$



Clifford Algebra and the Coxeter Plane – 4D case

rank 4	exponents	W-factorisation
A_4	$1,2,3,4$	$W = \exp\left(\frac{\pi}{5}B_C\right)\exp\left(\frac{2\pi}{5}IB_C\right)$
B_4	$1,3,5,7$	$W = \exp\left(\frac{\pi}{8}B_C\right)\exp\left(\frac{3\pi}{8}IB_C\right)$
D_4	$1,3,3,5$	$W = \exp\left(\frac{\pi}{6}B_C\right)\exp\left(\frac{\pi}{2}IB_C\right)$
F_4	$1,5,7,11$	$W = \exp\left(\frac{\pi}{12}B_C\right)\exp\left(\frac{5\pi}{12}IB_C\right)$
H_4	$1,11,19,29$	$W = \exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{11\pi}{30}IB_C\right)$

Actually, in 2, 3 and 4 dimensions it couldn't really be any other way

Clifford Algebra and the Coxeter Plane – D_6

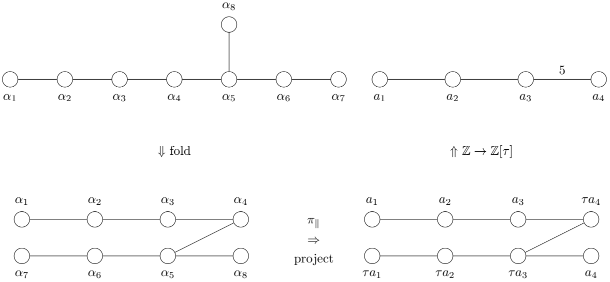
- For D_6 one has exponents $\boxed{1, 3, 5, 5, 7, 9}$
- Coxeter versor decomposes into orthogonal bits as

$$\boxed{W = \frac{1}{\sqrt{5}}(e_1 + e_2 + e_3 - e_4 - e_5)e_6 \exp\left(\frac{\pi}{10}B_C\right)\exp\left(\frac{3\pi}{10}B_2\right)}$$

- Now bivector exponentials correspond to rotations in orthogonal planes
- Vector factors correspond to reflections
- For odd n, there is always one such vector factor in D_n, and for even n there are two

Projection and Diagram Foldings

Compare with the 'partial folding' of E_8 to H_4



$$\boxed{s_{\beta_1} = s_{\alpha_1} s_{\alpha_7},\ s_{\beta_2} = s_{\alpha_2} s_{\alpha_6},\ s_{\beta_3} = s_{\alpha_3} s_{\alpha_5},\ s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4}$$

Imaginary differences – different imaginaries

So what has been gained by this Clifford view?

- There are different entities that serve as unit imaginaries
- They have a geometric interpretation as an eigenplane of the Coxeter element
- These don't need to commute with everything like i (though they do here – at least anticommute. But that is because we looked for orthogonal decompositions)
- But see that in general naive complexification can be a dangerous thing to do – unnecessary, issues of commutativity, confusing different imaginaries etc

Conformal geometry and Clifford algebra

- The conformal group $C(p,q) \sim SO(p+1,q+1)$
- So can use versor representation of conformal transformations in Clifford algebra (reflections, translations, inversions ...)
- Treat all of them multiplicatively in terms of versors and use sandwiching $\boxed{Ax\tilde{A}}$
- E.g. can generate a whole root lattice multiplicatively with compact reflection part and translations

Conformal Clifford Algebra

- The conformal group $C(n,p)$ is homomorphic to $Spin(n+1,p+1)$
- Go to $e_1, e_2, e_3, e, \bar{e}$, with $e_i^2 = 1, e^2 = 1, \bar{e}^2 = -1$
- Define two null vectors $n \equiv e + \bar{e}$, $\bar{n} \equiv e - \bar{e}$
- Can embed the 3D vector $x = x^\mu e_\mu = xe_1 + ye_2 + ze_3$ as a null vector in 5D ($\hat{X} \cdot n = -1$)

$$F(x) \equiv \hat{X} = \frac{1}{2\lambda^2}(x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

- Essentially linear action of $SO(n+1,p+1)$ in embedding space induces a non-linear realisation of the conformal group on the projective light cone (Dirac/Hestenes/Lasenby)
- So neat thing is that conformal transformations are now done by rotors (except inversion which is a reflection) – distances are given by inner products

Operations in Conformal Geometric Algebra

- Amsterdam protocol: $e = e_+$, $\bar{e} = e_-$, $n = n_\infty$ and $\bar{n} = n_0$.
- Reflections $\boxed{y' = -xyx}$ since e and $\bar{e} \Rightarrow n$ and $\bar{n}$ are orthogonal to $x \Rightarrow$ anticommute $-xnx = n$ and $-x\bar{n}x = \bar{n}$:

$$\boxed{-xF(y)x = F(y') = F(-xyx)}$$

- Rotations $\boxed{y' = Ry\tilde{R}}$ from reflections via Cartan-Dieudonné

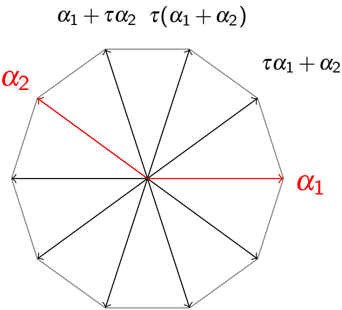
$$\boxed{RF(y)\tilde{R} = F(y') = F(Ry\tilde{R})}$$

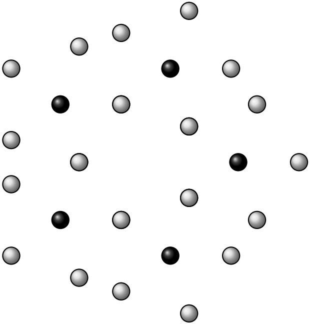
- Translations $\boxed{y' = y + a}$ rotor $\boxed{T_a = \exp\left(\frac{na}{2\lambda}\right) = 1 + \frac{na}{2\lambda}}$

$$\boxed{T_aF(y)\tilde{T}_a = F(y') = F(y + a)}$$

Proof of Principle

Construction of root systems and quasicrystalline point arrays carries through, e.g. here for H_2 and a pentagon with translation $1/\tau$





Benefits of this approach

- Conceptual Unification of Rotations and Translations via rotors
- Construct root system from the simple roots as before, and likewise for quasicrystalline point arrays
- Increased numerical stability (not really an issue here) due to projective representation

A new set of Bianchi IX Killing Vectors

- Used Conformal Clifford algebra setup to treat conformal group $C(1,3)$ as $SO(2,4)$
- Stabiliser subgroup of a certain vector gives the de Sitter group (Killing vectors)
- Using a certain projection broke this down to two commuting $SU(2) \times SU(2)$
- This is a new set of Bianchi IX Killing vectors with nice symmetry properties

A unified framework for polyhedral groups

Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm\tilde{A}xA$
$\mathrm{Spin}(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$\mathrm{Pin}(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the chiral icosahedral group has 60 elements, encoded in Clifford by 120 rotors, which form the binary icosahedral group
- together with the inversion/pseudoscalar I this gives 60 rotations and 60 rotoinversions, i.e. the full icosahedral group H_3 in 120 elements (with 240 versors)
- all three are interesting groups, e.g. in neutrino and flavour physics for family symmetry model building

Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate conjugacy classes etc of versors in Clifford
- Chiral (binary) polyhedral groups have irreps
- tetrahedral (12/24): 1, $1'$, $1''$, 2_s, $2'_s$, $2''_s$, 3
- octahedral (24/48): 1, $1'$, 2, 2_s, $2'_s$, 3, $3'$, 4_s
- icosahedral (60/120): 1, 2_s, $2'_s$, 3, $\overline{3}$, 4, 4_s, 5, 6_s
- All binary are discrete subgroups of $SU(2)$ and all thus have a 2_s spinor irrep
- See McKay correspondence
- Interesting to look at spinors/binary groups in their own right – see Induction Theorem

Some Group Theory: chiral, full, binary, pin

- Full (Coxeter) is just two copies of this (24/48/120 i.e. same order as binary since both $\mathrm{Spin}(3)$ and $O(3)$ are double covers of $SO(3)$)
- Pin group is just $1+I$ of this for B_3 and H_3, which contain the inversion I
- but not for A_3! (which doesn't – c.f. quaternionic reps)
- Instead $\mathrm{Pin}(A_3)$ has the same conjugacy classes as $\mathrm{Spin}(B_3)$

Conjugacy Classes: Quaternion group Q

- Five conjugacy classes: $\{1\}$, $\{-1\}$, $\{\pm e_1 e_2\}$, $\{\pm e_2 e_3\}$, $\{\pm e_3 e_1\}$
- Different conjugacy classes correspond to different geometric subspaces in the Clifford algebra
- Bit trivial for the quaternion group, but extends to arbitrary dimension

Conjugacy Classes: Binary octahedral group $2O$

- Eight conjugacy classes: $\{1\}$, $\{-1\}$, $\underline{6}$: bivectors $\{\pm e_1e_2, \pm e_2e_3, \pm e_3e_1\}$; $\underline{6}'$: bivector exponentials $\exp\underline{6}$; $\underline{6}''$: $\exp^{-}\underline{6}$; $\underline{8}$: spinors $\{1 \pm e_1e_2 \pm e_2e_3 \pm e_3e_1, \dots\}$; $\underline{8}'$: $-\underline{8}$; $\underline{12}$: bivectors $\{e_1(e_2+e_3), \dots\}$;
- Turns out most of these are the same as for $\mathrm{Pin}(A_3)$, and the remaining ones can be mapped to each other
- Though in $\mathrm{Pin}(A_3)$ also have odd grade elements, so some of the conjugacy classes are vector+trivector etc, i.e. different geometric interpretation

Character tables: Quaternion group Q (from $A_1^3 \to A_1^4$)

Q	1	-1	$\pm i$	$\pm j$	$\pm k$
1	1	1	1	1	1
$1'$	1	1	1	-1	-1
$1''$	1	1	-1	1	-1
$1'''$	1	1	-1	-1	1
2	2	-2	0	0	0
4	4	-4	0	0	0

Latter is of quaternionic type – somehow seen as particularly noteworthy

Character tables: binary octahedral group $2O$ (from $B_3 \to F_4$)

$2O$	1	1	6	8	8	6	6	12
1	1	1	1	1	1	1	1	1
$1'$	1	1	1	1	1	-1	-1	-1
2	2	2	2	-1	-1	0	0	0
3	3	3	-1	0	0	1	1	-1
$3'$	3	3	-1	0	0	-1	-1	1
4	4	-4	0	2	-2	$2\sqrt{2}$	$-2\sqrt{2}$	0
$4'$	4	-4	0	2	-2	$-2\sqrt{2}$	$2\sqrt{2}$	0
8	8	-8	0	-2	2	0	0	0

Again some of quaternionic type

Representations

- This Clifford multivector construction of the polyhedral groups is a faithful realisation/representation, i.e. is essentially the same as the abstract group
- But can define several different representations from these versor groups (may or may not be irreducible ones)
- Representations: matrices $D(R)$ such that

$$\boxed{D(R_1R_2) = D(R_1)D(R_2)}$$

Representations

- Representations: matrices $D(R)$ such that

$$\boxed{D(R_1R_2) = D(R_1)D(R_2)}$$

- Trivial representation: $D(R) = R1\tilde{R} = 1$
- Rotation representation: for nD vector $x = \sum a_i e_i$: $\boxed{D(R)\underline{x} = Rx\tilde{R}}$ usual $SO(n)$ $n \times n$-matrix
- Full representation: for nD vector $x = \sum a_i e_i$: $\boxed{D(A)\underline{x} = Ax\tilde{A}}$ usual $O(n)$ $n \times n$-matrix
- Spinor representation: for nD spinor y (2^{n-1} components): $\boxed{D(R)\underline{y} = Ry}$ a $2^{n-1} \times 2^{n-1}$-matrix
- Versor representation: for nD versor z (2^n components): $\boxed{D(A)\underline{z} = Az}$ a $2^n \times 2^n$-matrix

Character tables and Clifford reps: quaternion group Q

The spinor representation $\boxed{D(R)\underline{y} = R\underline{y}}$ of the quaternion group Q gives the representation of quaternionic type. (The trace of $D(R)$ is the character.)

Again just seen to be a consequence of the accidental isomorphism between 3D spinors and quaternions.

Q	1	-1	$\pm i$	$\pm j$	$\pm k$
1	1	1	1	1	1
$1'$	1	1	1	-1	-1
$1''$	1	1	-1	1	-1
$1'''$	1	1	-1	-1	1
2	2	-2	0	0	0
4	4	-4	0	0	0

Character tables and Clifford reps: binary octahedral group $2O$

The spinor representation $\boxed{D(R)\underline{y} = R\underline{y}}$ of the quaternion group $2O$ gives the irrep of quaternionic type.

The rotation representation $\boxed{D(R)\underline{x} = R\underline{x}\tilde{R}}$ gives $\mathbf{3}$ irrep.

$2O$	1	1	6	8	8	6	6	12
1	1	1	1	1	1	1	1	1
$1'$	1	1	1	1	1	-1	-1	-1
2	2	2	2	-1	-1	0	0	0
3	3	3	-1	0	0	1	1	-1
$3'$	3	3	-1	0	0	-1	-1	1
4	4	-4	0	2	-2	$2\sqrt{2}$	$-2\sqrt{2}$	0
$4'$	4	-4	0	2	-2	$-2\sqrt{2}$	$2\sqrt{2}$	0
8	8	-8	0	-2	2	0	0	0

Clifford: groups and representations summary

- Clifford algebra provides a unified framework for chiral/full/binary/pin polyhedral groups all in the same representation/realisation space/algebra (c.f. usual $SO(3)$ vs $SU(2)$ representations)
- Structure of the algebra $\Rightarrow$ different conjugacy classes are different kinds of objects in the algebra and are kept separate
- Several representations follow, in particular have geometric insight into complex and quaternionic representations

Clifford summary

- Interesting induction theorem linking geometry of 3D and 4D
- Geometric complex structures and non-trivial commutativity properties
- Simple versor representation of orthogonal transformations
- Conformal geometry
- Some interesting new results on group and representation theory
- Lie algebras can be constructed in Clifford algebra as bivector algebras, and Lie groups as spin groups (work with Phoenix)

Overview

Monstrous Moonshine

- Mysterious connection between two very different areas of Mathematics
- Modular forms (functions that live on a torus with complex structure τ): Fourier expansion coefficients wrt ($q = e^{2\pi i\tau}$)
- Finite simple groups: dimensions of irreducible representations
- Monstrous Moonshine: The largest sporadic group, the Monster M ($\{1, 196883, 21296876, \dots\}$) and the Klein $j(\tau)$ modular function
- $\boxed{j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots}$
- $196884 = 196883 + 1$, $21493760 = 21296876 + 196883 + 1$, $\dots$

Mathieu Moonshine

- Similar Moonshine phenomenon
- Modular form: elliptic genus of an $\mathcal{N}=4$ SCFT compactified on a K3-surface
- Finite simple group: Mathieu M_{24} $(\{45, 231, 770, 2277, 5796\dots\})$
- Elliptic genus is $\boxed{E_{K3}(\tau,z) = -2Ch(0;\tau,z) + 20Ch(1/2;\tau,z) + e(q)Ch(\tau,z)}$ where all the coefficients in the q-series $\boxed{e(q) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + \dots}$ are twice the dimension of some M_{24} irrep

Clifford and Moonshine

- Looking at Wess-Zumino-Witten models, i.e. strings propagating on a Lie group manifold
- Condition of extended supersymmetry ultimately hinges on classification of Clifford algebras – deep connection
- Connections with binary polyhedral groups, Monstrous Moonshine, McKay correspondence, lattices, affine extensions, Lie groups/algebras etc
- Elliptic genus is constant on (connected components of) the moduli space: better understanding as a topological feature as the index of a Dirac operator?

Thank you!

Back to the roots

- Unifying principles of geometry and symmetry
- Discrete groups (finite simple groups, polyhedral groups) and continuous groups (string compactifications, non-linear σ-models)
- Exceptional phenomena: E_8, $H_4/F_4/D_4$ (spinorial), McKay correspondence, Monster M, Mathieu M_{24} ...
- Applications: from mundane (viruses, fullerenes, quasicrystals) to exotic (HEP, Moonshine) – same mathematical tools: Coxeter, Clifford, affine extensions etc

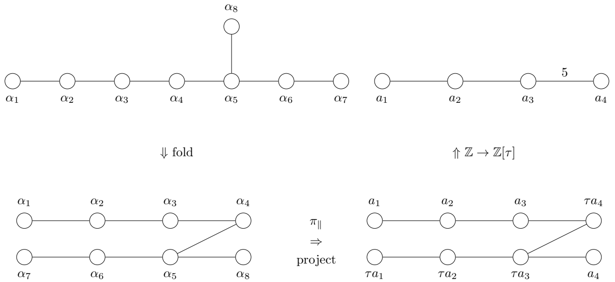
Mathematical Aspects: Better understanding of geometrical structures

- Affine extensions and translations – lattices, quasicrystals and projections
- Coxeter groups and Kac-Moody theory
- Euclidean, spherical, hyperbolic and conformal geometry
- Clifford and spinorial geometry
- Group theory, Lie groups and algebras (with Phoenix)
- Mathieu Moonshine, McKay correspondence
- Implications for the real world

HEP

- Gravitational and cosmological singularities – hidden symmetries
- New uses for non-crystallographic groups
- Topological defects
- Integrable systems
- Family symmetries (flavour/neutrino physics)
- 4D groups: A_4, B_4, D_4 – F_4, H_4?
- Spinor geometry
- E_8 features prominently – relation with H_4

Projection and Diagram Foldings



$$\boxed{s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}\,,\ \ s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}\,,\ \ s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}\,,\ \ s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4}$$

E_8 has two H_4-invariant subspaces – blockdiagonal form

D_6 has two H_3-invariant subspaces

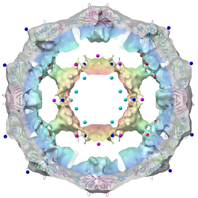
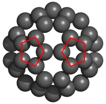
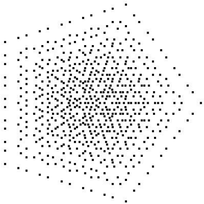
A_4 has two H_2-invariant subspaces

4D geometry is surprisingly important for HEP

- 4D root systems are surprisingly relevant to HEP
- A_4 is $SU(5)$ and comes up in Grand Unification
- D_4 is $SO(8)$ and is the little group of String theory
- In particular, its triality symmetry is crucial for showing the equivalence of RNS and GS strings
- B_4 is $SO(9)$ and is the little group of M-Theory
- F_4 is the largest crystallographic symmetry in 4D and H_4 is the largest non-crystallographic group
- The above are subgroups of the latter two
- Spinorial nature of the root systems could have surprising consequences for HEP

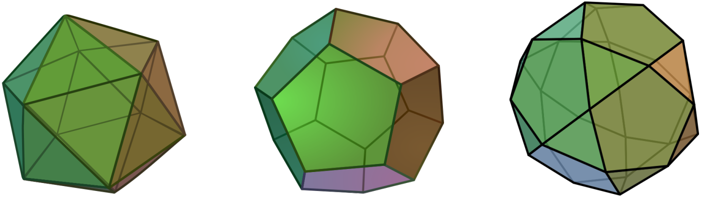
Motivation: Viruses

- Geometry of polyhedra described by Coxeter groups
- Viruses have to be 'economical' with their genes
- Encode structure modulo symmetry
- Largest discrete symmetry of space is the icosahedral group
- Many other 'maximally symmetric' objects in nature are also icosahedral: Fullerenes & Quasicrystals
- But: viruses are not just polyhedral – they have radial structure. Affine extensions give translations

Extend icosahedral group with distinguished translations

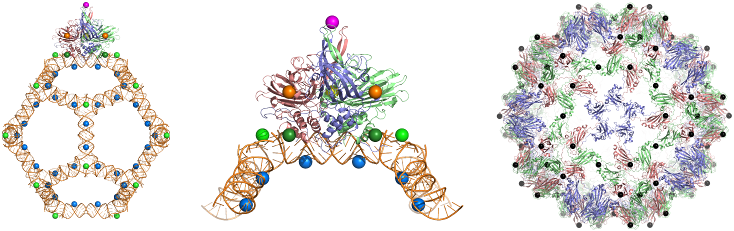
- Radial layers are simultaneously constrained by affine symmetry
- Works very well in practice: finite library of blueprints
- Select blueprint from the outer shape (capsid)
- Can predict inner structure (nucleic acid distribution) of the virus from the point array



Affine extensions of the icosahedral group (giving translations) and their classification.

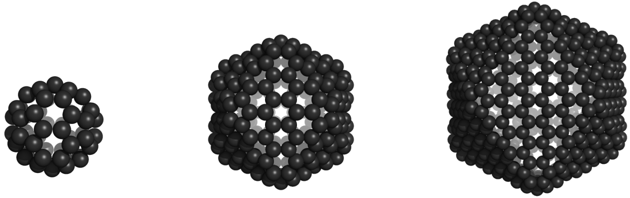
Use in Mathematical Virology

- Suffice to say point arrays work very exceedingly well in practice. Two papers on the mathematical (Coxeter) aspects.
- Implemented computational problem in Clifford – some very interesting mathematics comes out as well (see later).



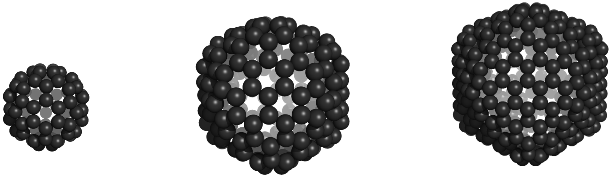
Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{60} - C_{240} - C_{540}$)



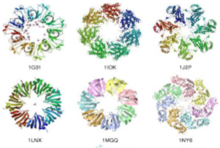
Extension to fullerenes: carbon onions

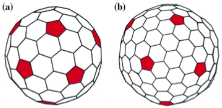
- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions $(C_{80} - C_{180} - C_{320})$

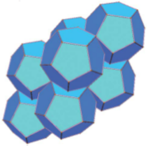


Applied areas (EPSRC proposal)

- Viruses: Extend to large viruses; interesting results on higher-order translations
- Proteins: Extend affine symmetry to $2D$ and apply to (chiral) proteins
- Fullerenes: Extend to larger fullerenes, in particular chiral carbon onions
- Packings: Novel analytical and numerical approaches to packings of polyhedral solids; with Colapinto (Santa Barbara), Twarock (York) and Thorpe (Phoenix)







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