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# What Clifford algebra can do for Coxeter groups and root systems

Pierre-Philippe Dechant

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Durham Mathematics HEP seminar – February 21, 2014

# The general theme: Geometry & Symmetry and their Applications

- Worked on a few different things: HEP – strings, particles and cosmology, pure maths and mathematical biology and Clifford algebras and mathematical physics
- Unifying themes of symmetry and geometry (euclidean, conformal, hyperbolic, spherical)
- Continuous Lie groups, e.g. for modeling cosmological spacetimes (Bianchi models)
- Discrete Coxeter groups and Kac-Moody algebras describe gravitational singularities/hidden symmetries in HEP theory, viruses, fullerenes, &c
- Mathematical frameworks of Coxeter groups and Clifford algebras

## 1 Introduction

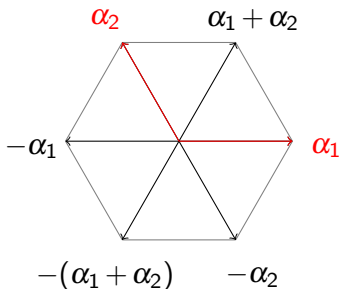
- Coxeter groups and root systems
- Clifford algebras

## 2 Coxeter and Clifford

- The Induction Theorem – from 3D to 4D
- The Coxeter Plane
- Conformal Geometry
- Some Group Theory

## 3 Moonshine and Outlook

## Root systems – $A_2$

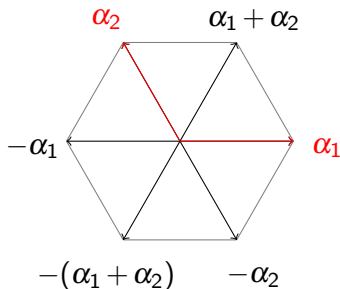


**Root system**  $\Phi$ : set of  
vectors  $\alpha$  such that

$$\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$$

and  $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

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$$\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$$

and  $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

**Simple roots**: express every element of  $\Phi$  via a  **$\mathbb{Z}$ -linear combination** (with coefficients of the same sign).

# Cartan Matrices

Cartan matrix of  $\alpha_i$ s is

$$A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2 \frac{|\alpha_j|}{|\alpha_i|} \cos \theta_{ij}$$

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$$\cos^2 \theta_{ij} = \frac{1}{4} A_{ij} A_{ji} \quad l_j^2 = \frac{A_{ij}}{A_{ji}} l_i^2$$

$$A_{ii} = 2 \quad A_{ij} \in \mathbb{Z}^{\leq 0} \quad A_{ij} = 0 \Leftrightarrow A_{ji} = 0.$$

$$A_2: A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

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Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal, simple link = roots at  $\frac{\pi}{3}$ , link with label  $m$  = angle  $\frac{\pi}{m}$ .

$$A_2 \circ \text{---} \circ$$

$$H_2 \circ \text{---}^5 \circ$$

$$I_2(n) \circ \text{---}^n \circ$$

# Coxeter groups

A **Coxeter group** is a group generated by some **involutive generators**  $s_i, s_j \in S$  subject to relations of the form  $(s_i s_j)^{m_{ij}} = 1$  with  $m_{ij} = m_{ji} \geq 2$  for  $i \neq j$ .

# Coxeter groups

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The **finite** Coxeter groups have a **geometric representation** where the involutions are realised as **reflections** at **hyperplanes through the origin** in a Euclidean vector space  $\mathcal{E}$ . In particular, let  $(\cdot|\cdot)$  denote the inner product in  $\mathcal{E}$ , and  $v, \alpha \in \mathcal{E}$ .

The **generator**  $s_\alpha$  corresponds to the **reflection**

$$s_\alpha : v \rightarrow s_\alpha(v) = v - 2 \frac{(v|\alpha)}{(\alpha|\alpha)} \alpha$$

at a hyperplane perpendicular to the **root vector**  $\alpha$ .

The action of the **Coxeter group** is to permute these **root vectors**.

# Coxeter groups vs Lie groups vs Lie algebras vs root systems

- Lie group = **group** and **manifold**
- Lie algebra = **bilinear, antisymmetric bracket** and **Jacobi identity**
- Lie algebras are **infinitesimal** version of Lie group = near the identity
- Can be more comprehensive e.g. 2D conformal algebra vs 2D conformal group
- But finite group transformation laws can be easier than linearising
- 'Nice' Lie algebras have **triangular decomposition**:

$$\mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$$

# Coxeter groups vs Lie groups vs Lie algebras vs root systems

- 'Nice' Lie algebras have **triangular decomposition**:  
 $\mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$
- $\mathcal{H}$  is the **Cartan subalgebra** (maximal commuting = quantum numbers)
- Creation and annihilation algebras  $\mathcal{N}$  form **root lattice**
- Symmetry group is called **Weyl group** and is a **crystallographic Coxeter group**:  $A_n, B_n/C_n, D_n, G_2, F_4, E_6, E_7, E_8$
- So Coxeter groups in theoretical physics always crystallographic! **Neglect**  $I_2(n), H_3, H_4$ .
- Useful Lie algebras are **(semi-)simple** LA (determinant of Cartan matrix is positive), **affine** LA (determinant is 0), **Kac-Moody algebras**, Borcherd's algebras...

# Kac-Moody algebras

- Kac-Moody algebras  $\mathcal{A}$  of rank  $N$  are defined by **generalised Cartan** ( $N \times N$ ) matrices with  $A_{ii} = 2$ ,  $A_{ij} \in \mathbb{Z}_-$  ( $i \neq j$ ) and  $A_{ij} \neq 0 \Rightarrow A_{ji} \neq 0$
- $3N$  **generators**  $h_i, e_i, f_i$  satisfy **Chevalley-Serre relations**  
 $[h_i, h_j] = 0$   $[h_i, e_j] = A_{ij}e_j$ ,  $[h_i, f_j] = -A_{ij}f_j$ ,  $[e_i, f_j] = \delta_{ij}h_i$   
 $\underbrace{[e_i, [e_i, [\dots, [e_i, e_j]]]] \dots]}_{1-A_{ij} \text{ times}} = 0$ ,  $\underbrace{[f_i, [f_i, [\dots, [f_i, f_j]]]] \dots]}_{1-A_{ij} \text{ times}} = 0$
- **Simple roots**  $\alpha_i$  are  $[h, e_i] = \alpha_i(h)e_i$

## Example – $A_1$ , $SU(2)$ , Angular Momentum



- Cartan subalgebra = Quantum number:  $L_z$
- $\mathcal{N}_+$ : raising operator  $L_+ = \alpha$
- $\mathcal{N}_-$ : lowering operator  $L_- = -\alpha$
- ( $L^2$  is Casimir/commutes with all algebra elements, is however not actually in the algebra!)

## Example – $A_1$ , $SU(2)$ , Electroweak

$W^+$



$W^-$

- Cartan subalgebra – Quantum number:  $A$
- $\mathcal{N}_+$ : raising operator  $W^+ = \alpha$
- $\mathcal{N}_-$ : lowering operator  $W^- = -\alpha$
- (Since SM electroweak is actually  $SU(2) \times U(1)$ ,  $U(1)$  gives another field  $i$ , such that physical  $Z^0$  and  $\gamma$  are superpositions of  $A$  and  $i$ )
- Also  $W^\pm$  now charged and self-interact, unlike QED

## Affine extensions

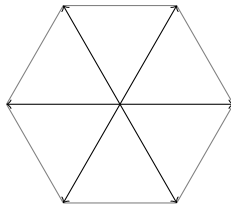
An **affine Coxeter group** is the extension of a Coxeter group by an **affine reflection in a hyperplane not containing the origin**  $s_{\alpha_0}^{aff}$  whose geometric action is given by

$$s_{\alpha_0}^{aff} v = \alpha_0 + v - \frac{2(\alpha_0 | v)}{(\alpha_0 | \alpha_0)} \alpha_0$$

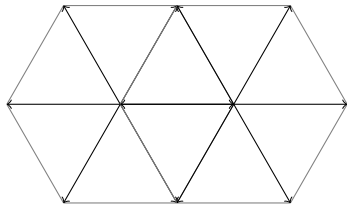
**Non-distance preserving:** includes the **translation generator**

$$Tv = v + \alpha_0 = s_{\alpha_0}^{aff} s_{\alpha_0} v$$

# Affine extensions – $A_2$

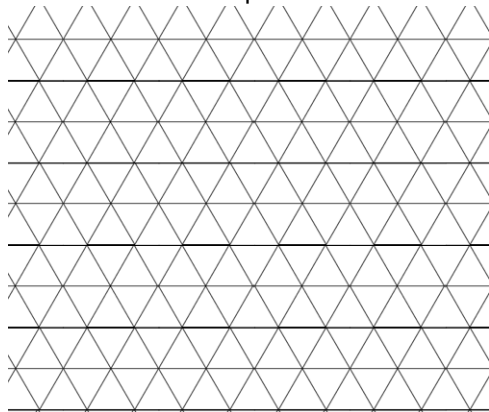


## Affine extensions – $A_2$

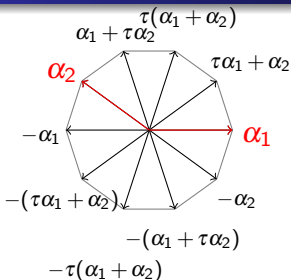


## Affine extensions – $A_2$

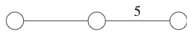
Affine extensions of crystallographic Coxeter groups lead to a **tessellation** of the plane and a **lattice**.



# Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$



$$A = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$ : 10, 120, 14,400 elements, the only Coxeter groups that generate rotational symmetries of order 5  
linear combinations now in the extended integer ring

$$\mathbb{Z}[\tau] = \{a + \tau b \mid a, b \in \mathbb{Z}\}$$

golden ratio

$$\tau = \frac{1}{2}(1 + \sqrt{5}) = 2 \cos \frac{\pi}{5}$$

$$x^2 = x + 1$$

$$\tau' = \sigma = \frac{1}{2}(1 - \sqrt{5}) = 2 \cos \frac{2\pi}{5}$$

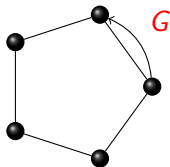
$$\tau + \sigma = 1, \tau\sigma = -1$$

## What's new?

- In HEP, mostly come from **Lie groups**, then **Lie algebras**, then their **Weyl groups** and **root systems**
- This only gives the **crystallographic Coxeter** groups
- Do the **non-crystallographic** Coxeter groups have something interesting to offer? In particular, **affine extensions**?
- Interesting connections between the **geometries of different dimensions**: Relation between **crystallographic and non-crystallographic** ( $E_8$  and  $H_4$ ) and my **spinor construction** (3 & 4D)
- Both could have **interesting consequences for HEP** (4D groups and  $E_8$  feature heavily) and other applications (**viruses, quasicrystals, proteins, fullerenes...**)

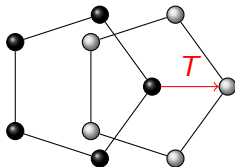
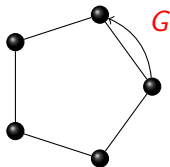
# Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon



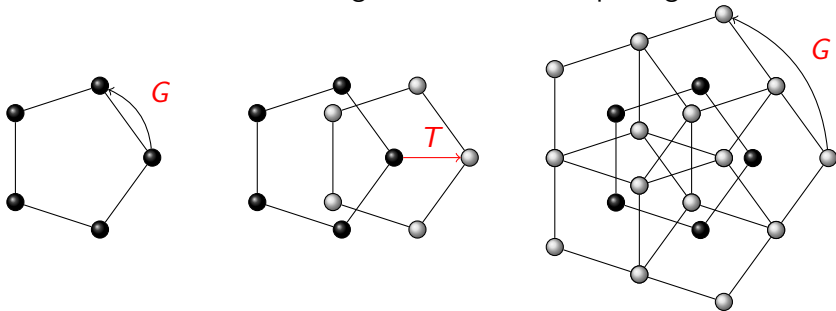
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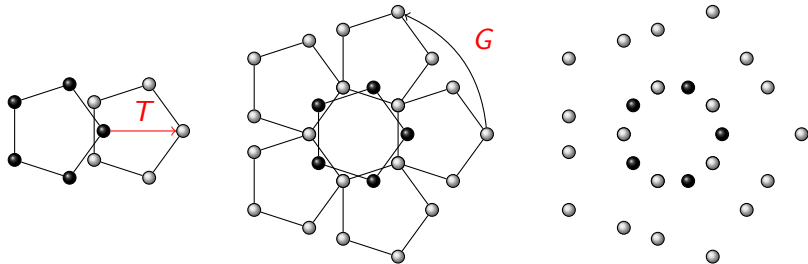
Unit translation along a vertex of a unit pentagon



A **random** translation would give 5 secondary pentagons, i.e. 25 points. Here we have **degeneracies** due to '**coinciding points**'.

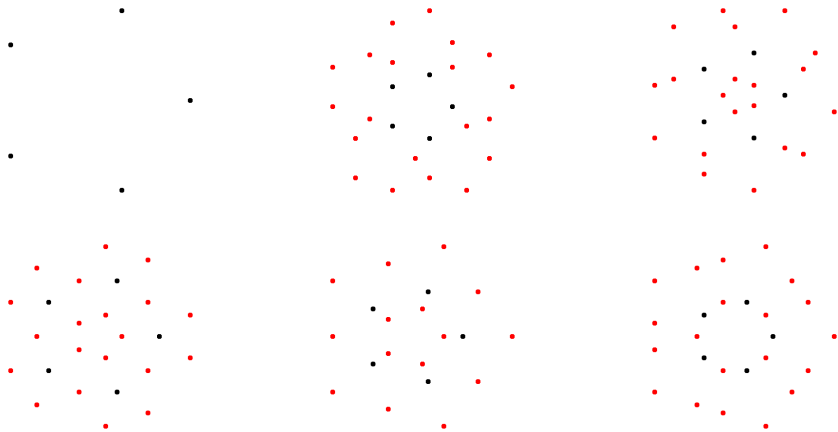
# Affine extensions of non-crystallographic root systems

Translation of length  $\tau = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$  (golden ratio)

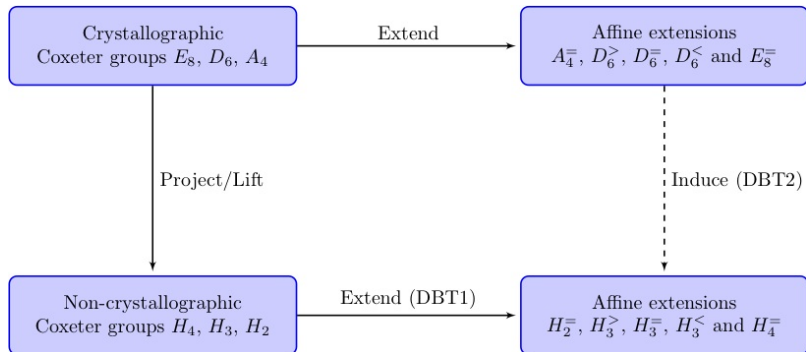


Looks like a **virus** or **carbon onion**

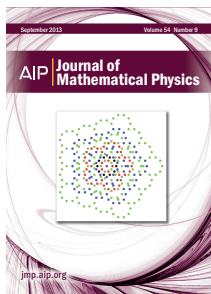
# More Blueprints



# Road Map



# Applications of affine extensions of non-crystallographic root systems



There are interesting applications to **quasicrystals**, **viruses** or **carbon onions**, but here concentrate on the **mathematical** aspects

# Basics of Clifford Algebra I

- Form an algebra using the **Geometric Product**  
 $ab \equiv a \cdot b + a \wedge b$  for two vectors
- Extend via linearity and associativity to higher grade elements (**multivectors**)
- For an  $n$ -dimensional space generated by  $n$  orthogonal unit vectors  $e_i$  have  $2^n$  elements
- Then  $e_i e_j = e_i \wedge e_j = -e_j e_i$  so **anticommute** (Grassmann variables, exterior algebra)
- Unlike the inner and outer products separately, this product is **invertible**
- This feeds through to the **differential structure** of the theory with more powerful **Greens functions methods**  $\nabla^{-1}$

## Basics of Clifford Algebra II

- These are known to have **matrix representations** over the normed division algebras  $\mathbb{R}$ ,  $\mathbb{C}$  and  $\mathbb{H}$   $\Rightarrow$  **Classification** of Clifford algebras
- E.g. **Pauli algebra** in 3D (likewise for **Dirac algebra** in 4D) is

$$\begin{array}{cccc}
 \underbrace{\{1\}} & \underbrace{\{e_1, e_2, e_3\}} & \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}} & \underbrace{\{I \equiv e_1 e_2 e_3\}} \\
 1 \text{ scalar} & 3 \text{ vectors} & 3 \text{ bivectors} & 1 \text{ trivector}
 \end{array}$$

- These have the well-known matrix representations in terms of  **$\sigma$ - and  $\gamma$ -matrices**
- Working with these is not necessarily the most insightful thing to do, so here stress approach to **work directly** with the algebra
- Naturally have things that **square to  $-1$** , e.g.

$$\boxed{(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1}, \text{ and } \text{non-trivial} \\
 \text{commutation properties}$$

## Reflections

- Clifford algebra is **very efficient** at performing **reflections**
- Consider reflecting the vector  $a$  in a hypersurface with unit normal  $n$ :

$$a' = a_{\perp} - a_{\parallel} = a - 2a_{\parallel} = a - 2(a \cdot n)n$$

- c.f. **fundamental Weyl reflection**  $s_i : v \rightarrow s_i(v) = v - 2 \frac{(v|\alpha_i)}{(\alpha_i|\alpha_i)} \alpha_i$
- But in Clifford algebra have  $n \cdot a = \frac{1}{2}(na + an)$  so reassembles into **sandwiching**

$$a' = -nan$$

- So both **Coxeter** and **Clifford** frameworks are ideally suited to describing **reflections** – first to combine the two

## Reflections and Rotations

- Generate a **rotation** when compounding two reflections wrt  $n$  then  $m$  (**Cartan-Dieudonné theorem**):

$$a'' = mnanm \equiv Ra\tilde{R}$$

where  $R = mn$  is called a **rotor** and a tilde denotes **reversal** of the order of the constituent vectors ( $R\tilde{R} = 1$ )

- Now neat thing is all multivectors transform **covariantly** e.g.

$$MN \rightarrow (RM\tilde{R})(RN\tilde{R}) = RM\tilde{R}RN\tilde{R} = R(MN)\tilde{R}$$

so transform **double-sidedly**

- Rotors form a **group**, the rotor group, which gives a representation of the **Spin group**  $Spin(n)$  – they transform **single-sidedly** (obvious now it's a double (universal) cover)

# Artin's Theorem and orthogonal transformations

- **Artin**: every isometry is at most  $d$  reflections
- Since have a **double cover** of reflections ( $n$  and  $-n$ ) we have a **double cover** of  $O(p, q)$ :  $\text{Pin}(p, q)$

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1$$

- Pinors/versors = products of vectors  $n_1 n_2 \dots n_k$  encode orthogonal transformations via '**sandwiching**'
- **Cartan-Dieudonné**: rotations are an even number of reflections:  $\text{Spin}(p, q)$  doubly covers  $SO(p, q)$
- The conformal group  $C(p, q) \sim SO(p+1, q+1)$  so can use these for **translations, inversions** etc as well

## Spinor techniques

- Of course there is a **matrix representation**  $\underline{R}$  for the action of a **spinor**:  $\underline{R}x = Rx\tilde{R}$
- This is the usual **rotation matrix**  $\underline{R}$  in  $SO(p, q)$
- Having the **spin double cover/square root of the rotation matrix** can be convenient
- E.g. can get **differential equations for spinor**  $R$  that are **easier** to solve, then can reconstitute  $\underline{R}$  if necessary

## 1 Introduction

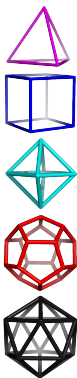
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## 3 Moonshine and Outlook

## 3D Platonic Solids



- There are 5 Platonic solids
- Tetrahedron (**self-dual**) ( $A_3$ )
- **Dual** pair **octahedron** and **cube** ( $B_3$ )
- **Dual** pair **icosahedron** and **dodecahedron** ( $H_3$ )
- Only the **octahedron** is a **root system** (actually for  $(A_1^3)$ )

# Clifford and Coxeter: Platonic Solids



Platonic Solid	Group	root system
Tetrahedron	$A_3$ $A_1^3$	Cuboctahedron Octahedron
Octahedron Cube	$B_3$	Cuboctahedron +Octahedron
Icosahedron Dodecahedron	$H_3$	Icosidodecahedron

- **Platonic Solids** have been known for millennia

# Clifford and Coxeter: Platonic Solids



$A_1^3$

$A_3$

$B_3$

$H_3$

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- Described by **Coxeter** groups

# Clifford and Coxeter: Platonic Solids



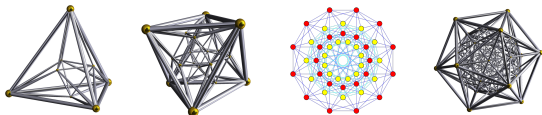
$A_1^3$   $A_1^4$

$A_3$   $D_4$

$B_3$   $F_4$

$H_3$   $H_4$

- **Platonic Solids** have been known for millennia; described by **Coxeter** groups
- Concatenating reflections gives **Clifford** spinors (**binary polyhedral groups**)
- These **induce 4D root systems**  
 $\psi = a_0 + a_i |e_i \Rightarrow \psi \tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- 4D analogues of the Platonic Solids and give rise to 4D **Coxeter** groups



## 4D 'Platonic Solids'

- In 4D, there are **6 analogues** of the Platonic Solids:
- **5-cell** (self-dual) ( $A_4$ )
- **24-cell** (self-dual) ( $D_4$ ) – a 24-cell and its dual together are the  $F_4$  root system
- Dual pair **16-cell** and **8-cell** ( $B_4$ )
- Dual pair **600-cell** and **120-cell** ( $H_4$ )
- **24-cell, 16-cell and 600-cell** are all **root systems**, as is the related  $F_4$  root system
- 8-cell and 120-cell are dual to a root system, so in 4D **out of 6 Platonic Solids only the 5-cell** (corresponding to  $A_n$  family) is not related to a root system!
- The 4D Platonic solids are **not normally thought to be related to the 3D ones** except for the boundary cells

# Spinorial Symmetries of 4D Polytopes

## Spinorial symmetries

rank 3	$ \Phi $	$ W $	rank 4	$ \Phi $	Symmetry
$A_3$	12	24	$D_4$ 24-cell	24	$2 \cdot 24^2 = 576$
$B_3$	18	48	$F_4$ lattice	48	$48^2 = 2304$
$H_3$	30	120	$H_4$ 600-cell	120	$120^2 = 14400$
$A_1^3$	6	8	$A_1^4$ 16-cell	8	$3! \cdot 8^2 = 384$
$A_1 \oplus A_2$	8	12	$A_2 \oplus A_2$ prism	12	$12^2 = 144$
$A_1 \oplus H_2$	12	20	$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$A_1 \oplus I_2(n)$	$n+2$	$2n$	$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

Similar for **Grand Antiprism** ( $H_4$  without  $H_2 \oplus H_2$ ) and **Snub 24-cell** ( $2I$  without  $2T$ ).

# Induction Theorem

- Theorem: **3D spinor groups** are **root systems** ( $R$  and  $-R$  are in a spinor group by construction, and closure under reflections is guaranteed by the **closure property of the spinor group**)
- Induction Theorem: **Every rank-3 root system induces a rank-4 root system.**
- Counterexample: **not every rank-4 root system** is induced in this way
- Spinor group is trivially **closed** under **conjugation, left and right multiplication**. Results in **non-trivial symmetries** when viewed as a **polytope/root system**.
- Now explains **symmetry** of the polytopes/root system and thus the **order** of the rank-4 Coxeter group

# Induction Theorem

- So induced **4D polytopes** are actually **root systems**.
- Clear why the **number of roots**  $|\Phi|$  is equal to  $|G|$ , the **order of the spinor group**
- Theorem: The **automorphism group** of the induced root system contains **two factors** of the respective spinor group acting from the **left** and the **right**.
- Only **remaining cases** in 3D are  $A_1 \oplus I_2(n)$ , which give  $I_2(n) \oplus I_2(n)$

## General Case of Induction

Only **remaining case** is what happens for  $A_1 \oplus I_2(n)$  - this gives a **doubling**  $I_2(n) \oplus I_2(n)$

rank 3	rank 4
$A_3$	$D_4$
$B_3$	$F_4$
$H_3$	$H_4$
$A_1^3$	$A_1^4$
$A_1 \oplus A_2$	$A_2 \oplus A_2$
$A_1 \oplus H_2$	$H_2 \oplus H_2$
$A_1 \oplus I_2(n)$	$I_2(n) \oplus I_2(n)$

# Spinorial Symmetries of 4D Polytopes

## Spinorial symmetries

rank 3	$ \Phi $	$ W $	rank 4	$ \Phi $	Symmetry
$A_3$	12	24	$D_4$ 24-cell	24	$2 \cdot 24^2 = 576$
$B_3$	18	48	$F_4$ lattice	48	$48^2 = 2304$
$H_3$	30	120	$H_4$ 600-cell	120	$120^2 = 14400$
$A_1^3$	6	8	$A_1^4$ 16-cell	8	$3! \cdot 8^2 = 384$
$A_1 \oplus A_2$	8	12	$A_2 \oplus A_2$ prism	12	$12^2 = 144$
$A_1 \oplus H_2$	12	20	$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$A_1 \oplus I_2(n)$	$n+2$	$2n$	$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

Similar for **Grand Antiprism** ( $H_4$  without  $H_2 \oplus H_2$ ) and **Snub 24-cell** ( $2I$  without  $2T$ ). Additional factors in the automorphism group come from **3D Dynkin diagram symmetries!**

## Some non-Platonic examples of spinorial symmetries

- **Grand Antiprism**: the **100** vertices achieved by subtracting 20 vertices of  $H_2 \oplus H_2$  from the 120 vertices of the  $H_4$  root system 600-cell – two separate orbits of  $H_2 \oplus H_2$
- This is a semi-regular polytope with automorphism symmetry  $\text{Aut}(H_2 \oplus H_2)$  of order  $400 = 20^2$
- Think of the  $H_2 \oplus H_2$  as coming from the **doubling procedure?** (Likewise for  $\text{Aut}(A_2 \oplus A_2)$  subgroup)
- **Snub 24-cell**:  $2T$  is a subgroup of  $2I$  so subtracting the 24 corresponding vertices of the 24-cell from the 600-cell, one gets a semiregular polytope with **96** vertices and automorphism group  $2T \times 2T$  of order  $576 = 24^2$ .

## Sub root systems

- The above spinor groups had spinor multiplication as the **group operation**
- But also closed under **twisted conjugation** – corresponds to **closure under reflections** (root system property)
- If we take **twisted conjugation** as the group operation instead, we can have various **subgroups**
- These are the remaining **4D root systems** e.g.  $A_4$  or  $B_4$

## What's new?

- Novel **connection** between geometry of **3D and 4D**
- In fact, 3D seems more **fundamental** – contrary to the **usual perspective** of 3D subgroups of 4D groups
- **Spinorial symmetries**
- Clear why **spinor group** gives a root system and why **two factors** of the same group reappear in the **automorphism group**
- Novel **spinorial perspective** on 4D geometry
- **Accidentalness** of the spinor construction and **exceptional** 4D phenomena
- Connection with Arnold's **trinitities**, the **McKay correspondence** and **Monstrous Moonshine**

## Recap: Clifford algebra and reflections & rotations

- Clifford algebra is **very efficient** at performing **reflections** via **sandwiching**

$$a' = -nan$$

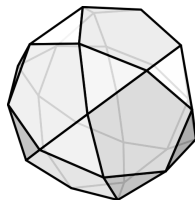
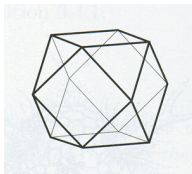
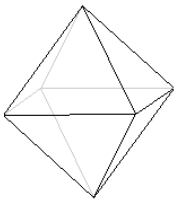
- Generate a **rotation** when compounding two reflections wrt  $n$  then  $m$  (**Cartan-Dieudonné theorem**):

$$a'' = mnanm \equiv Ra\tilde{R}$$

where  $R = mn$  is called a **rotor** and a tilde denotes **reversal** of the order of the constituent vectors ( $R\tilde{R} = 1$ )

## From the Coxeter simple roots to the root system

- Take the  $A_1 \times A_1 \times A_1$  simple roots  $(1,0,0)$ ,  $(0,1,0)$ ,  $(0,0,1)$   
 $\Rightarrow$  under reflections get  $(-1,0,0)$ ,  $(0,-1,0)$ ,  $(0,0,-1)$ , the vertices of an **octahedron**.
- Take the **three simple roots** of  $A_1 \times A_1 \times A_1/A_3/B_3/H_3$ .  
Closure under Clifford **reflections** generate the whole root system of 6/12/18/30 **vertices of an octahedron/cuboctahedron/ cuboctahedron with an octahedron/ icosidodecahedron**).



## Spinors from reflections

- These are the 3D Coxeter groups that are symmetry groups of the **Platonic Solids** (tetrahedron and octahedron are similar but simpler than the icosahedron)
- The 6/12/18/30 **reflections** in  $A_1 \times A_1 \times A_1 / A_3 / B_3 / H_3$  generate 8/24/48/120 **rotors**.
- E.g.  $(\pm 1, 0, 0)$ ,  $(0, \pm 1, 0)$ ,  $(0, 0, \pm 1)$  give the 8 permutations of  $(\pm 1; 0, 0, 0)$  (scalar and bivector parts, the notation will become clear later).
- The **discrete spinor group** is isomorphic to the **quaternion group**  $Q$  / **binary tetrahedral group**  $2T$  / **binary octahedral group**  $2O$  / **binary icosahedral group**  $2I$ ).

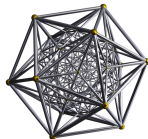
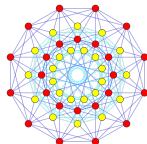
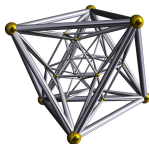
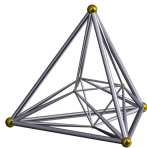
## A unified framework for polyhedral groups

Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm \tilde{A}xA$
$Spin(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$Pin(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the **chiral icosahedral** group has 60 elements, encoded in Clifford by 120 rotors, which form the **binary icosahedral** group
- together with the **inversion/pseudoscalar**  $I$  this gives 60 rotations and 60 rotoinversions, i.e. the **full icosahedral** group  $H_3$  in 120 elements (with 240 versors)
- all three are interesting groups, e.g. in **neutrino and flavour physics** for family symmetry model building

# Spinors and Polytopes

- The space of  $Cl(3)$ -spinors and quaternions have a **4D Euclidean signature**:  $\psi = a_0 + a_i l e_i \Rightarrow \psi \tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- Can reinterpret **spinors in  $\mathbb{R}^3$**  as **vectors in  $\mathbb{R}^4$**
- Then the spinors constitute the **vertices** of the **16-cell**, **24-cell**, **24-cell and dual 24-cell** and the **600-cell**
- These are 4D analogues of the **Platonic Solids**: regular convex 4-polytopes



## Spinors, Polytopes and Root systems

- The 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell are in fact the root systems of  $A_1 \times A_1 \times A_1 \times A_1$ ,  $D_4$ ,  $F_4$  and  $H_4$
- Exceptional phenomena:  $D_4$  (triality, important in string theory),  $F_4$  (largest lattice symmetry in 4D),  $H_4$  (largest non-crystallographic symmetry)
- Exceptional  $D_4$  and  $F_4$  arise from series  $A_3$  and  $B_3$
- In fact, can strengthen this statement on inducing polytopes to statement on inducing root systems

## Root systems in three and four dimensions

The **spinors** generated from the reflections contained in the respective **rank-3 Coxeter group** via the geometric product are realisations of the **binary polyhedral groups**  $Q$ ,  $2T$ ,  $2O$  and  $2I$ , which were known to generate (mostly exceptional) **rank-4 groups**, but **not known why**, and why the '**mysterious symmetries**'.

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$		$Q$	$A_1 \times A_1 \times A_1 \times A_1$	
$A_3$		$2T$	$D_4$	
$B_3$		$2O$	$F_4$	
$H_3$		$2I$	$H_4$	

# Arnold's Trinities

Arnold's observation that many areas of real mathematics can be **complexified** and **quaternionified** resulting in theories with a similar structure.

- The **fundamental trinity** is thus  $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The **projective spaces**  $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The **spheres**  $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The **Möbius/Hopf bundles**  $(S^1 \rightarrow S^1, S^4 \rightarrow S^2, S^7 \rightarrow S^4)$
- The **Lie Algebras**  $(E_6, E_7, E_8)$
- The symmetries of the **Platonic Solids**  $(A_3, B_3, H_3)$
- The **4D groups**  $(D_4, F_4, H_4)$
- **New connections** via my **Clifford spinor construction** (see McKay correspondence)

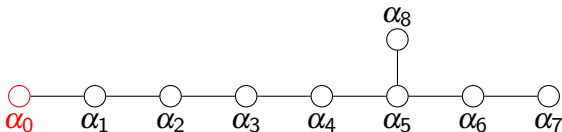
## Platonic Trinities

- Arnold's connection between  $(A_3, B_3, H_3)$  and  $(D_4, F_4, H_4)$  is **very convoluted** and involves numerous other trinities at intermediate steps:
- **Decomposition of the projective plane** into Weyl chambers and Springer cones
- The **number of Weyl chambers** in each segment is  $24 = 2(1 + 3 + 3 + 5)$ ,  $48 = 2(1 + 5 + 7 + 11)$ ,  $120 = 2(1 + 11 + 19 + 29)$
- Notice this miraculously **matches the quasihomogeneous weights**  $((2, 4, 4, 6), (2, 6, 8, 12), (2, 12, 20, 30))$  of the Coxeter groups  $(D_4, F_4, H_4)$
- Believe the Clifford connection is **more direct**

## Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate **conjugacy classes** etc of versors in Clifford
- Chiral (**binary**) polyhedral groups have irreps
- tetrahedral (12/24): 1, 1', 1'', 2<sub>s</sub>, 2'<sub>s</sub>, 2''<sub>s</sub>, 3
- octahedral (24/48): 1, 1', 2, 2<sub>s</sub>, 2'<sub>s</sub>, 3, 3', 4<sub>s</sub>
- icosahedral (60/120): 1, 2<sub>s</sub>, 2'<sub>s</sub>, 3,  $\bar{3}$ , 4, 4<sub>s</sub>, 5, 6<sub>s</sub>
- All binary are **discrete subgroups of  $SU(2)$**  and all thus have a 2<sub>s</sub> spinor irrep
- Connection with the **McKay correspondence!**

# Affine extensions – $E_8^=$



$$-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$$

AKA  $E_8^+$  and along with  $E_8^{++}$  and  $E_8^{+++}$  thought to be the underlying symmetry of **String and M-theory**

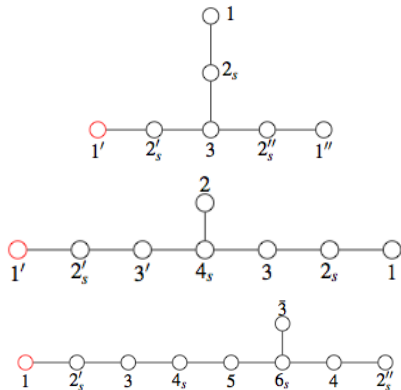
Also interesting from a pure mathematics point of view:  **$E_8$  lattice**, **McKay correspondence** and **Monstrous Moonshine**.

# The McKay Correspondence

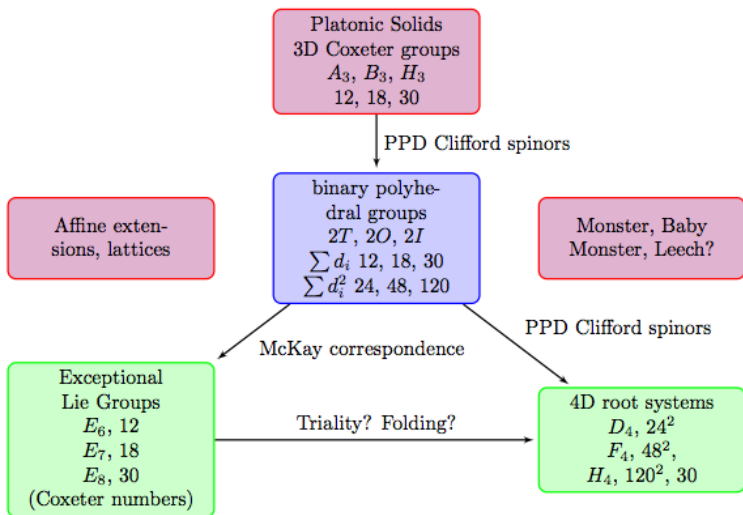
binary polyhedral groups  
 $2T, 2O, 2I$   
 $\sum d_i$  12, 18, 30  
 $\sum d_i^2$  24, 48, 120

McKay correspondence

Exceptional Lie Groups  
 $E_6, 12$   
 $E_7, 18$   
 $E_8, 30$   
 (Coxeter numbers)



# The McKay Correspondence



# The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the **cyclic** and **dicyclic groups** are in correspondence with  $A_n$  and  $D_n$ , e.g. the quaternion group  $Q$  and  $D_4^+$ . So McKay correspondence not just a trinity but **ADE-classification**. We also have  $I_2(n)$  on top of the trinity ( $A_3, B_3, H_3$ )

rank-3 group	diagram	binary	rank-4 group	diagram	Lie algebra	diagram
$A_1 \times A_1 \times A_1$		$Q$	$A_1 \times A_1 \times A_1 \times A_1$		$D_3^+$	
$A_3$		$2T$	$D_4$		$E_6^+$	
$B_3$		$2O$	$F_4$		$E_7^+$	
$H_3$		$2I$	$H_4$		$E_8^+$	

## 4D geometry is surprisingly important for HEP

- 4D root systems are **surprisingly relevant to HEP**
- $A_4$  is  $SU(5)$  and comes up in **Grand Unification**
- $D_4$  is  $SO(8)$  and is the little group of **String theory**
- In particular, its **triality symmetry** is crucial for showing the equivalence of RNS and GS strings
- $B_4$  is  $SO(9)$  and is the little group of **M-Theory**
- $F_4$  is the **largest crystallographic** symmetry in 4D and  $H_4$  is the **largest non-crystallographic** group
- The above are **subgroups** of the latter two
- **Spinorial nature** of the root systems could have **surprising consequences for HEP**

# Quaternions and Clifford Algebra

- The unit **spinors**  $\{1; i e_1; i e_2; i e_3\}$  of  $Cl(3)$  are isomorphic to the **quaternion** algebra  $\mathbb{H}$  (up to sign)
- The 3D **Hodge dual of a vector** is a **pure bivector** which corresponds to a **pure quaternion**, and their products are identical (up to sign)

## Discrete Quaternion groups

- The 8 quaternions of the form  $(\pm 1, 0, 0, 0)$  and permutations are called the **Lipschitz units**, and form a realisation of the **quaternion group** in 8 elements.
- The 8 Lipschitz units together with  $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$  are called the **Hurwitz units**, and realise the **binary tetrahedral group** of order 24. Together with the 24 'dual' quaternions of the form  $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$ , they form a group isomorphic to the **binary octahedral group** of order 48.
- The 24 Hurwitz units together with the 96 unit quaternions of the form  $(0, \pm \tau, \pm 1, \pm \sigma)$  and even permutations, are called the **Icosians**. The icosian group is isomorphic to the **binary icosahedral group** with 120 elements.

## Quaternionic representations of 3D and 4D Coxeter groups

- Groups  $E_8$ ,  $D_4$ ,  $F_4$  and  $H_4$  have representations in terms of **quaternions**
- **Extensively used** in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why
- e.g.  $H_4$  consists of 120 elements of the form  $(\pm 1, 0, 0, 0)$ ,  $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$  and  $(0, \pm \tau, \pm 1, \pm \sigma)$
- Seen as remarkable that the **subset of the 30 pure quaternions** is a realisation of  $H_3$  (**a sub-root system**)
- Similarly,  $A_3$ ,  $B_3$ ,  $A_1 \times A_1 \times A_1$  have representations in terms of **pure quaternions**
- Will see there is a **much simpler geometric explanation**

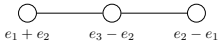
# Quaternionic representations used in the literature



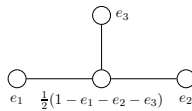
$A_1 \times A_1 \times A_1$



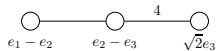
$A_1 \times A_1 \times A_1 \times A_1$



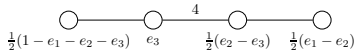
$A_3 = D_3$



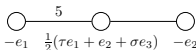
$D_4$



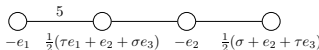
$B_3$



$F_4$



$H_3$



$H_4$

# Demystifying Quaternionic Representations

- 3D: **Pure quaternions** = Hodge dualised (pseudoscalar) **root vectors**
- In fact, they are the **simple roots of the Coxeter groups**
- 4D: **Quaternions** = disguised **spinors** – but those of the **3D Coxeter group** i.e. the binary polyhedral groups!
- This relation between 3D and 4D via the geometric product does not seem to be known
- Quaternion multiplication = ordinary Clifford reflections and rotations

# Demystifying Quaternionic Representations

- **Pure quaternion subset** of 4D groups only gives 3D group if the 3D group **contains the inversion/pseudoscalar  $I$**
- e.g. **does not work** for the tetrahedral group  $A_3$ , but  $A_3 \rightarrow D_4$  **induction still works**, with the central node essentially 'spinorial'
- In fact, it goes the other way around: the **3D groups induce the 4D groups** via spinors
- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as  **$R_1 = \alpha_1 \alpha_2$  and  $R_2 = \alpha_2 \alpha_3$**
- Can see these are '**spinor generators**' and how they don't really contain any more information/roots than the rank-3 groups alone

## Quaternions vs Clifford versors

- **Sandwiching** is often seen as particularly nice feature of the **quaternions giving rotations**
- This is actually a **general feature** of Clifford algebras/versors **in any dimension**; the isomorphism to the **quaternions** is **accidental** to 3D
- However, the **root system** construction does not necessarily generalise
- 2D generalisation merely gives that  $I_2(n)$  is **self-dual**
- **Octonionic** generalisation just induces two copies of the above 4D root systems, e.g.  $A_3 \rightarrow D_4 \oplus D_4$

## References (single-author)

- Clifford algebra unveils a surprising geometric significance of quaternionic root systems of Coxeter groups  
Advances in Applied Clifford Algebras, June 2013, Volume 23, Issue 2, pp 301-321
- A Clifford algebraic framework for Coxeter group theoretic computations (Conference Prize at AGACSE 2012)  
Advances in Applied Clifford Algebras (2013)
- Nomination for W.K. Clifford Prize (2014)
- Invitation to Arizona State University
- Rank-3 root systems induce root systems of rank 4 via a new Clifford spinor construction arXiv:1207.7339 (2012)
- Platonic Solids generate their 4-dimensional analogues  
Acta Cryst. A69 (2013)

## Coxeter Elements, Degrees and Exponents

- Like the symmetric group, Coxeter groups can have **invariant polynomials**. Their **degrees**  $d$  are important invariants/group characteristics.
- Turns out that actually **degrees**  $d$  are intimately related to so-called **exponents**  $m$   $m = d - 1$ .

## Coxeter Elements, Degrees and Exponents

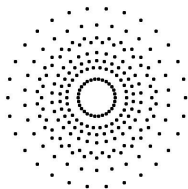
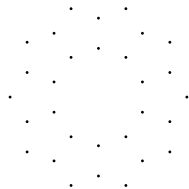
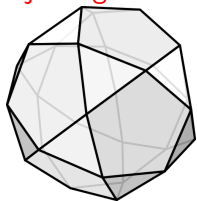
- A **Coxeter Element** is any combination of all the simple reflections  $w = s_1 \dots s_n$ , i.e. in Clifford algebra it is encoded by the versor  $W = \alpha_1 \dots \alpha_n$  acting as  $v \rightarrow wv = \pm \tilde{W} v W$ . All such elements are conjugate and thus their **order** is invariant and called the **Coxeter number  $h$** .
- The Coxeter element has **complex eigenvalues** of the form  $\exp(2\pi mi/h)$  where  $m$  are called **exponents**.
- Standard theory **complexifies** the real Coxeter group situation in order to find **complex eigenvalues**, then takes **real sections** again (the unfortunate standard procedure in many situations) – without any insight into the complex structure (or in fact, there are different ones).

## Coxeter Elements, Degrees and Exponents

- The Coxeter element has **complex eigenvalues** of the form  $\exp(2\pi mi/h)$  where  $m$  are called **exponents**
- Standard theory **complexifies** the real Coxeter group situation in order to find **complex eigenvalues**, then takes **real** sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure(s)
- In particular, **1** and  **$h-1$**  are always exponents
- Turns out that actually **exponents and degrees** are intimately related ( $m = d - 1$ ). The construction is slightly roundabout but uniform, and uses the **Coxeter plane**.

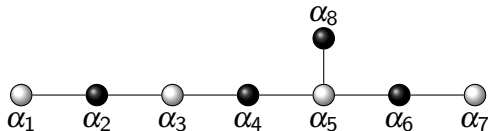
# The Coxeter Plane

- Can show **every** (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by **projecting** their root system onto the Coxeter plane



# The Coxeter Plane

- Obvious from Clifford point of view, that Coxeter element has eigenspaces (**eigenblades**) rather than just eigenvectors
- In particular, can show **every** (for our purposes) Coxeter group has a Coxeter plane
- Existence relies on the fact that all groups in question have **tree-like Dynkin diagrams**, and thus admit an **alternate colouring**
- Essentially just gives **two sets of mutually commuting generators**



# The Coxeter Plane

- Existence relies on the fact that all groups in question have **tree-like Dynkin diagrams**, and thus admit an alternate colouring
- Essentially just gives **two sets of orthogonal = mutually commuting generators but anticommuting root vectors**  $\alpha_w$  and  $\alpha_b$  (duals  $\omega$ )
- Cartan matrices are positive definite, and thus have a **Perron-Frobenius** (all positive) eigenvector  $\lambda_j$ .
- Take **linear combinations** of components of this eigenvector as coefficients of two vectors from the orthogonal sets  

$$v_w = \sum \lambda_w \omega_w \text{ and } v_b = \sum \lambda_b \omega_b$$
- Their **outer product/Coxeter plane bivector**  $B_C = v_b \wedge v_w$  describes an **invariant plane** where  $w$  acts by rotation by  $2\pi/h$ .

## Clifford Algebra and the Coxeter Plane – 2D case

- For  $I_2(n)$  take  $\alpha_1 = e_1, \alpha_2 = -\cos \frac{\pi}{n} e_1 + \sin \frac{\pi}{n} e_2$

- So **Coxeter versor** is just

$$W = \alpha_1 \alpha_2 = -\cos \frac{\pi}{n} + \sin \frac{\pi}{n} e_1 e_2 = -\cos \frac{\pi}{n} + \sin \frac{\pi}{n} I = -\exp\left(-\frac{\pi I}{n}\right)$$

- In Clifford algebra it is therefore immediately obvious that the action of the  $I_2(n)$  Coxeter element is described by a versor (here a rotor/spinor) that encodes **rotations in the  $e_1 e_2$ -Coxeter-plane** and yields  $h = n$  since trivially  $W^n = (-1)^{n+1}$  yielding  $w^n = 1$  via  $wv = \tilde{W}vW$ .

## Clifford Algebra and the Coxeter Plane – 2D case

- So **Coxeter versor** is just  $W = -\exp\left(-\frac{\pi I}{n}\right)$
- $I = e_1 e_2$  **anticommutes** with both  $e_1$  and  $e_2$  such that **sandwiching formula** becomes

$$v \rightarrow wv = \tilde{W}vW = \tilde{W}^2v = \exp\left(\pm\frac{2\pi I}{n}\right)v \text{ immediately}$$

yielding the standard result for the **complex eigenvalues** in real Clifford algebra **without any need for artificial complexification**

- The Coxeter plane bivector  $B_C = e_1 e_2 = I$  gives the **complex structure**
- The Coxeter plane bivector  $B_C$  is invariant under the **Coxeter versor**  $\tilde{W}B_CW = \pm B_C$ .

## Clifford Algebra and the Coxeter Plane – 3D case

- In 3D,  $A_3$ ,  $B_3$ ,  $H_3$  have  $\{1, 2, 3\}$ ,  $\{1, 3, 5\}$  and  $\{1, 5, 9\}$
- Coxeter element is product of a **spinor** in the Coxeter plane with the same complex structure as before, and a **reflection perpendicular** to the plane
- So in 3D still completely determined by the plane
- $1$  and  $h-1$  are **rotations** in **Coxeter plane**
- $h/2$  is the **reflection** (for  $v$  in the normal direction)

$$wv = \tilde{W}^2 = \exp\left(\pm \frac{2\pi I h}{h} \frac{h}{2}\right) = \exp(\pm \pi I)v = -v$$

## Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can **factorise**  $W$  into **orthogonal** (commuting/anticommuting) components  

$$W = \alpha_1 \dots \alpha_n = W_1 \dots W_n$$
 with  $W_i = \exp(\pi m_i l_i / h)$
- Here,  $l_i$  is a bivector describing a **plane** with  $l_i^2 = -1$
- For  $v$  **orthogonal to the plane** described by  $l_i$  we have  

$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i W_i v = v$$
 so cancels out
- For  $v$  **in the plane** we have  

$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i l_i / h) v$$
- Thus if we **decompose  $W$  into orthogonal eigenspaces**, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

## Clifford algebra: no need for complexification

- For  $v$  in the plane we have

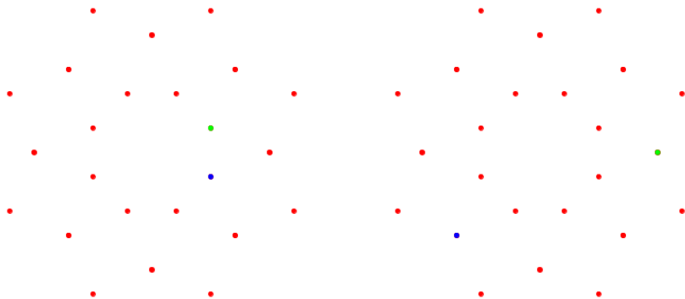
$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i l_i / h) v$$

- So **complex eigenvalue equation** arises geometrically **without any need** for complexification
- **Different complex structures** immediately give different **eigenplanes**
- Eigenvalues/angles/**exponents** given from just factorising  $W = \alpha_1 \dots \alpha_n$
- E.g.  $B_4$  has exponents 1, 3, 5, 7 and  $W = \exp\left(\frac{\pi}{8} l_1\right) \exp\left(\frac{3\pi}{8} l_2\right)$
- Here we have been looking for orthogonal eigenspaces, so **innocuous** – different complex structures commute
- But not in general – **naive complexification** can be misleading

# Clifford Algebra and the Coxeter Plane – 4D case

- E.g.  $B_4$  has exponents 1, 3, 5, 7
- Coxeter versor decomposes into **orthogonal components**

$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} I B_C\right)$$



# Clifford Algebra and the Coxeter Plane – 4D case

rank 4	exponents	W-factorisation
$A_4$	1, 2, 3, 4	$W = \exp\left(\frac{\pi}{5} B_C\right) \exp\left(\frac{2\pi}{5} I B_C\right)$
$B_4$	1, 3, 5, 7	$W = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} I B_C\right)$
$D_4$	1, 3, 3, 5	$W = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{\pi}{2} I B_C\right)$
$F_4$	1, 5, 7, 11	$W = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} I B_C\right)$
$H_4$	1, 11, 19, 29	$W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$

Actually, in 2, 3 and 4 dimensions it couldn't really be any other way

## Clifford Algebra and the Coxeter Plane – $D_6$

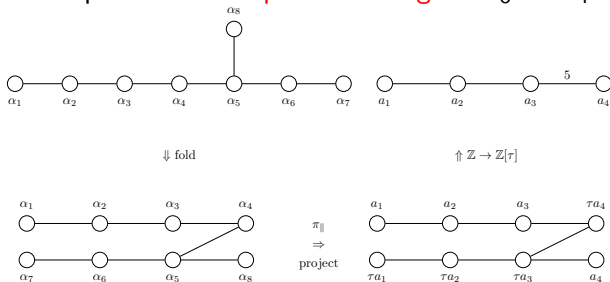
- For  $D_6$  one has exponents  $1, 3, 5, 5, 7, 9$
- Coxeter versor decomposes into orthogonal bits as

$$W = \frac{1}{\sqrt{5}}(e_1 + e_2 + e_3 - e_4 - e_5)e_6 \exp\left(\frac{\pi}{10}B_C\right) \exp\left(\frac{3\pi}{10}B_2\right)$$

- Now **bivector exponentials** correspond to **rotations in orthogonal planes**
- **Vector** factors correspond to **reflections**
- For odd  $n$ , there is always **one such vector factor** in  $D_n$ , and for even  $n$  there are **two**

# Projection and Diagram Foldings

Compare with the 'partial folding' of  $E_8$  to  $H_4$



$$S_{\beta_1} = S_{\alpha_1} S_{\alpha_7}, \quad S_{\beta_2} = S_{\alpha_2} S_{\alpha_6}, \quad S_{\beta_3} = S_{\alpha_3} S_{\alpha_5}, \quad S_{\beta_4} = S_{\alpha_4} S_{\alpha_8} \Rightarrow H_4$$

## Imaginary differences – different imaginaries

So what has been **gained** by this **Clifford view**?

- There are **different** entities that serve as **unit imaginaries**
- They have a **geometric** interpretation as an **eigenplane of the Coxeter element**
- These don't need to **commute** with everything like  $i$  (though they do here – at least anticommute. But that is because we looked for **orthogonal decompositions**)
- But see that in general **naive complexification** can be a dangerous thing to do – **unnecessary**, issues of **commutativity**, **confusing** different imaginaries etc

# Conformal geometry and Clifford algebra

- The conformal group  $C(p, q) \sim SO(p+1, q+1)$
- So can use **versor representation** of conformal transformations in Clifford algebra (**reflections, translations, inversions ...**)
- Treat all of them **multiplicatively** in terms of versors and use **sandwiching**  $Ax\tilde{A}$
- E.g. can generate a whole **root lattice** multiplicatively with **compact reflection part** and **translations**

## Conformal Clifford Algebra

- The **conformal group**  $C(n, p)$  is homomorphic to  $Spin(n+1, p+1)$
- Go to  $e_1, e_2, e_3, e, \bar{e}$ , with  $e_i^2 = 1, e^2 = 1, \bar{e}^2 = -1$
- Define two null vectors  $n \equiv e + \bar{e}, \bar{n} \equiv e - \bar{e}$
- Can embed the 3D vector  $x = x^\mu e_\mu = xe_1 + ye_2 + ze_3$  as a **null vector in 5D** ( $\hat{X} \cdot n = -1$ )

$$F(x) \equiv \hat{X} = \frac{1}{2\lambda^2}(x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

- Essentially **linear** action of  $SO(n+1, p+1)$  in embedding space induces a **non-linear** realisation of the conformal group on the **projective light cone** (Dirac/Hestenes/Lasenby)
- So neat thing is that **conformal transformations** are now done by **rotors** (except inversion which is a reflection) – distances are given by **inner products**

# Operations in Conformal Geometric Algebra

- **Amsterdam protocol:**  $e = e_+$ ,  $\bar{e} = e_-$ ,  $n = n_\infty$  and  $\bar{n} = n_0$ .
- **Reflections**  $y' = -xyx$  since  $e$  and  $\bar{e} \Rightarrow n$  and  $\bar{n}$  are orthogonal to  $x \Rightarrow$  anticommute  $-xnx = n$  and  $-x\bar{n}x = \bar{n}$ :

$$-xF(y)x = F(y') = F(-xyx)$$

- **Rotations**  $y' = Ry\tilde{R}$  from reflections via **Cartan-Dieudonné**

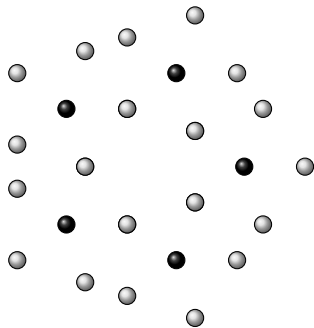
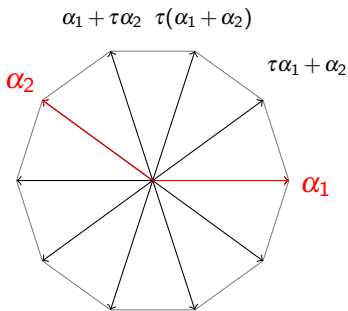
$$RF(y)\tilde{R} = F(y') = F(Ry\tilde{R})$$

- **Translations**  $y' = y + a$  rotor  $T_a = \exp\left(\frac{na}{2\lambda}\right) = 1 + \frac{na}{2\lambda}$

$$T_a F(y) \tilde{T}_a = F(y') = F(y + a)$$

# Proof of Principle

Construction of **root systems** and **quasicrystalline point arrays** carries through, e.g. here for  $H_2$  and a pentagon with translation  $1/\tau$



## Benefits of this approach

- **Conceptual Unification** of Rotations and Translations via rotors
- Construct **root system** from the simple roots as before, and likewise for **quasicrystalline point arrays**
- Increased **numerical stability** (not really an issue here) due to projective representation

## A new set of Bianchi IX Killing Vectors

- Used **Conformal Clifford algebra** setup to treat conformal group  $C(1,3)$  as  $SO(2,4)$
- Stabiliser subgroup of a certain vector gives the **de Sitter** group (Killing vectors)
- Using a certain **projection** broke this down to **two commuting  $SU(2) \times SU(2)$**
- This is a **new set of Bianchi IX Killing vectors** with **nice symmetry properties**

## A unified framework for polyhedral groups

Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm \tilde{A}xA$
$Spin(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$Pin(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the **chiral icosahedral** group has 60 elements, encoded in Clifford by 120 rotors, which form the **binary icosahedral** group
- together with the **inversion/pseudoscalar**  $I$  this gives 60 rotations and 60 rotoinversions, i.e. the **full icosahedral** group  $H_3$  in 120 elements (with 240 versors)
- all three are interesting groups, e.g. in **neutrino and flavour physics** for family symmetry model building

## Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate **conjugacy classes** etc of versors in Clifford
- Chiral (**binary**) polyhedral groups have irreps
- tetrahedral (12/24): 1, 1', 1'',  $2_s$ ,  $2'_s$ ,  $2''_s$ , 3
- octahedral (24/48): 1, 1', 2,  $2_s$ ,  $2'_s$ , 3, 3',  $4_s$
- icosahedral (60/120): 1,  $2_s$ ,  $2'_s$ , 3,  $\bar{3}$ , 4,  $4_s$ , 5,  $6_s$
- All binary are **discrete subgroups of  $SU(2)$**  and all thus have a  $2_s$  spinor irrep
- See **McKay correspondence**
- Interesting to look at spinors/binary groups in their own right – see **Induction Theorem**

## Some Group Theory: chiral, full, binary, pin

- Full (**Coxeter**) is just **two copies** of this (24/48/120 i.e. same order as binary since both  $\text{Spin}(3)$  and  $O(3)$  are double covers of  $SO(3)$ )
- **Pin group** is just  $1 + I$  of this for  $B_3$  and  $H_3$ , which contain the inversion  $I$
- but **not for  $A_3$ !** (which doesn't – c.f. quaternionic reps)
- Instead  $\text{Pin}(A_3)$  has the same conjugacy classes as  $\text{Spin}(B_3)$

## Conjugacy Classes: Quaternion group $Q$

- Five conjugacy classes:  $\{1\}$ ,  $\{-1\}$ ,  $\{\pm e_1 e_2\}$ ,  $\{\pm e_2 e_3\}$ ,  $\{\pm e_3 e_1\}$
- Different conjugacy classes correspond to different geometric subspaces in the Clifford algebra
- Bit trivial for the quaternion group, but extends to arbitrary dimension

## Conjugacy Classes: Binary octahedral group $2O$

- Eight conjugacy classes:  $\{1\}$ ,  $\{-1\}$ ,  $\underline{6}$ : **bivectors**  $\{\pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1\}$ ;  $\underline{6}'$ : **bivector exponentials**  $\exp^{\underline{6}}$ ;  $\underline{6}''$ :  $\exp^{-\underline{6}}$ ;  $\underline{8}$ : **spinors**  $\{1 \pm e_1 e_2 \pm e_2 e_3 \pm e_3 e_1, \dots\}$ ;  $\underline{8}'$ :  $-\underline{8}$ ;  $\underline{12}$ : **bivectors**  $\{e_1(e_2 + e_3), \dots\}$
- Turns out most of these are the **same as for  $\text{Pin}(A_3)$** , and the remaining ones can be mapped to each other
- Though in  $\text{Pin}(A_3)$  also have **odd grade** elements, so some of the conjugacy classes are vector+trivector etc, i.e. different geometric interpretation

# Character tables: Quaternion group $Q$ (from $A_1^3 \rightarrow A_1^4$ )

$Q$	1	-1	$\pm i$	$\pm j$	$\pm k$
1	1	1	1	1	1
1'	1	1	1	-1	-1
1''	1	1	-1	1	-1
1'''	1	1	-1	-1	1
2	2	-2	0	0	0
4	4	-4	0	0	0

Latter is of **quaternionic type** – somehow seen as particularly noteworthy

# Character tables: binary octahedral group $2O$ (from $B_3 \rightarrow F_4$ )

$2O$	1	1	6	8	8	6	6	12
1	1	1	1	1	1	1	1	1
$1'$	1	1	1	1	1	-1	-1	-1
2	2	2	2	-1	-1	0	0	0
3	3	3	-1	0	0	1	1	-1
$3'$	3	3	-1	0	0	-1	-1	1
4	4	-4	0	2	-2	$2\sqrt{2}$	$-2\sqrt{2}$	0
$4'$	4	-4	0	2	-2	$-2\sqrt{2}$	$2\sqrt{2}$	0
8	8	-8	0	-2	2	0	0	0

Again some of quaternionic type

# Representations

- This **Clifford multivector construction** of the polyhedral groups is a **faithful realisation/representation**, i.e. is essentially the same as the abstract group
- But can define several different **representations** from these versor groups (may or may not be **irreducible** ones)
- Representations: matrices  $D(R)$  such that

$$D(R_1 R_2) = D(R_1) D(R_2)$$

# Representations

- Representations: matrices  $D(R)$  such that

$$D(R_1 R_2) = D(R_1) D(R_2)$$

- **Trivial** representation:  $D(R) = R1\tilde{R} = 1$
- **Rotation** representation: for nD vector  $x = \sum a_i e_i$ :

$$D(R)\underline{x} = R x \tilde{R} \quad \text{usual } SO(n) \ n \times n\text{-matrix}$$

- Full representation: for nD vector  $x = \sum a_i e_i$ :  $D(A)\underline{x} = A x \tilde{A}$   
 usual  $O(n)$   $n \times n$ -matrix

- **Spinor** representation: for nD spinor  $y$  ( $2^{n-1}$  components):

$$D(R)\underline{y} = R y \quad \text{a } 2^{n-1} \times 2^{n-1}\text{-matrix}$$

- **Versor** representation: for nD versor  $z$  ( $2^n$  components):

$$D(A)\underline{z} = A z \quad \text{a } 2^n \times 2^n\text{-matrix}$$

Character tables and Clifford reps: quaternion group  $Q$ 

The **spinor** representation  $D(R)\underline{y} = R\underline{y}$  of the quaternion group  $Q$  gives the representation of **quaternionic type**. (The **trace** of  $D(R)$  is the **character**.)

Again just seen to be a consequence of the **accidental isomorphism** between 3D spinors and quaternions.

$Q$	1	-1	$\pm i$	$\pm j$	$\pm k$
1	1	1	1	1	1
1'	1	1	1	-1	-1
1''	1	1	-1	1	-1
1'''	1	1	-1	-1	1
2	2	-2	0	0	0
4	4	-4	0	0	0

# Character tables and Clifford reps: binary octahedral group $2O$

The **spinor** representation  $D(R)\underline{y} = Ry$  of the quaternion group  $2O$  gives the irrep of **quaternionic type**.

The **rotation** representation  $D(R)\underline{x} = Rx\tilde{R}$  gives **3 irrep**.

$2O$	1	1	6	8	8	6	6	12
1	1	1	1	1	1	1	1	1
$1'$	1	1	1	1	1	-1	-1	-1
2	2	2	2	-1	-1	0	0	0
3	3	3	-1	0	0	1	1	-1
$3'$	3	3	-1	0	0	-1	-1	1
4	4	-4	0	2	-2	$2\sqrt{2}$	$-2\sqrt{2}$	0
$4'$	4	-4	0	2	-2	$-2\sqrt{2}$	$2\sqrt{2}$	0
8	8	-8	0	-2	2	0	0	0

## Clifford: groups and representations summary

- Clifford algebra provides a **unified framework** for chiral/full/binary/pin **polyhedral** groups all in the **same** representation/realisation space/**algebra** (c.f. usual  $SO(3)$  vs  $SU(2)$  representations)
- Structure of the algebra  $\Rightarrow$  different **conjugacy classes** are **different kinds of objects in the algebra** and are kept separate
- Several **representations** follow, in particular have **geometric insight** into **complex and quaternionic** representations

## Clifford summary

- Interesting **induction theorem** linking geometry of 3D and 4D
- Geometric **complex structures** and **non-trivial commutativity** properties
- Simple versor representation of **orthogonal transformations**
- **Conformal** geometry
- Some interesting new results on **group and representation theory**
- **Lie algebras** can be constructed in Clifford algebra as **bivector algebras**, and **Lie groups** as **spin groups** (work with Phoenix)

## 1 Introduction

- Coxeter groups and root systems
- Clifford algebras

## 2 Coxeter and Clifford

- The Induction Theorem – from 3D to 4D
- The Coxeter Plane
- Conformal Geometry
- Some Group Theory

## 3 Moonshine and Outlook

# Monstrous Moonshine

- **Mysterious connection** between two very different areas of Mathematics
- **Modular forms** (functions that live on a torus with complex structure  $\tau$ ): Fourier expansion **coefficients** wrt ( $q = e^{2\pi i\tau}$ )
- **Finite simple groups**: **dimensions** of irreducible representations
- **Monstrous Moonshine**: The largest sporadic group, the **Monster  $M$**  ( $\{1, 196883, 21296876, \dots\}$ ) and the Klein  **$j(\tau)$**  modular function
- $j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots$
- $196884 = 196883 + 1, 21493760 = 21296876 + 196883 + 1, \dots$

# Mathieu Moonshine

- Similar Moonshine phenomenon
- Modular form: **elliptic genus** of an  $\mathcal{N} = 4$  SCFT compactified on a **K3-surface**
- Finite simple group: **Mathieu  $M_{24}$**   
 ( $\{45, 231, 770, 2277, 5796, \dots\}$ )
- Elliptic genus is

$$E_{K3}(\tau, z) = -2Ch(0; \tau, z) + 20Ch(1/2; \tau, z) + e(q)Ch(\tau, z)$$

where all the coefficients in the  $q$ -series

$$e(q) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + \dots$$

are twice the dimension of some  $M_{24}$  irrep

## Clifford and Moonshine

- Looking at **Wess-Zumino-Witten models**, i.e. strings propagating on a Lie group manifold
- Condition of **extended supersymmetry** ultimately hinges on **classification** of Clifford algebras – deep connection
- Connections with **binary polyhedral groups, Monstrous Moonshine, McKay correspondence, lattices, affine extensions, Lie groups/algebras** etc
- **Elliptic genus** is constant on (connected components of) the **moduli space**: better understanding as a **topological** feature as the **index of a Dirac operator**?

Thank you!

## Back to the roots

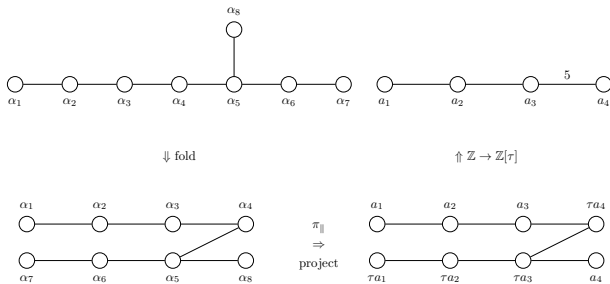
- Unifying principles of **geometry** and **symmetry**
- **Discrete** groups (finite simple groups, polyhedral groups) and **continuous** groups (string compactifications, non-linear  $\sigma$ -models)
- **Exceptional phenomena**:  $E_8$ ,  $H_4/F_4/D_4$  (spinorial), McKay correspondence, Monster  $M$ , Mathieu  $M_{24}$  ...
- **Applications**: from **mundane** (viruses, fullerenes, quasicrystals) to **exotic** (HEP, Moonshine) – **same mathematical tools**: Coxeter, Clifford, affine extensions etc

## Mathematical Aspects: Better understanding of geometrical structures

- **Affine extensions** and translations – lattices, quasicrystals and projections
- **Coxeter** groups and **Kac-Moody** theory
- Euclidean, spherical, hyperbolic and conformal geometry
- Clifford and **spinorial** geometry
- Group theory, Lie groups and algebras (with Phoenix)
- **Mathieu Moonshine**, McKay correspondence
- Implications for the **real world**

- Gravitational and cosmological singularities – hidden symmetries
- New uses for non-crystallographic groups
- Topological defects
- Integrable systems
- Family symmetries (flavour/neutrino physics)
- 4D groups:  $A_4$ ,  $B_4$ ,  $D_4 - F_4$ ,  $H_4$ ?
- Spinor geometry
- $E_8$  features prominently – relation with  $H_4$

# Projection and Diagram Foldings



$$s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}, \quad s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}, \quad s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}, \quad s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4$$

$E_8$  has two  $H_4$ -invariant subspaces – blockdiagonal form

$D_6$  has two  $H_3$ -invariant subspaces

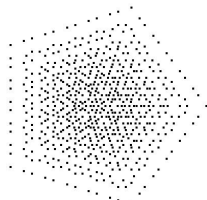
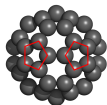
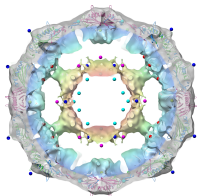
$A_4$  has two  $H_2$ -invariant subspaces

## 4D geometry is surprisingly important for HEP

- 4D root systems are **surprisingly relevant to HEP**
- $A_4$  is  $SU(5)$  and comes up in **Grand Unification**
- $D_4$  is  $SO(8)$  and is the little group of **String theory**
- In particular, its **triality symmetry** is crucial for showing the equivalence of RNS and GS strings
- $B_4$  is  $SO(9)$  and is the little group of **M-Theory**
- $F_4$  is the **largest crystallographic** symmetry in 4D and  $H_4$  is the **largest non-crystallographic** group
- The above are **subgroups** of the latter two
- **Spinorial nature** of the root systems could have **surprising consequences for HEP**

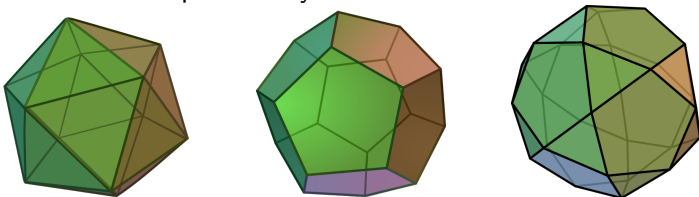
## Motivation: Viruses

- Geometry of **polyhedra** described by **Coxeter** groups
- Viruses have to be '**economical**' with their **genes**
- Encode **structure** modulo **symmetry**
- **Largest discrete symmetry of space** is the **icosahedral** group
- Many other '**maximally symmetric**' objects in nature are also icosahedral: **Fullerenes & Quasicrystals**
- But: viruses are not just polyhedral – they have **radial structure**. **Affine extensions** give **translations**



## Extend icosahedral group with distinguished translations

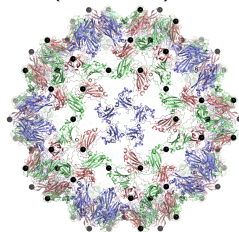
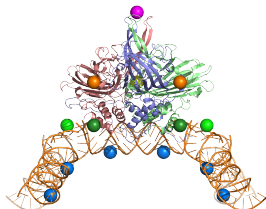
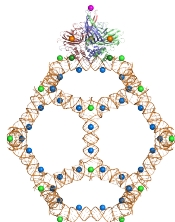
- Radial layers are **simultaneously constrained** by affine symmetry
- Works very well in practice: **finite library of blueprints**
- **Select** blueprint from the **outer shape** (capsid)
- Can **predict inner structure** (nucleic acid distribution) of the virus from the point array



**Affine extensions** of the icosahedral group (giving translations) and their **classification**.

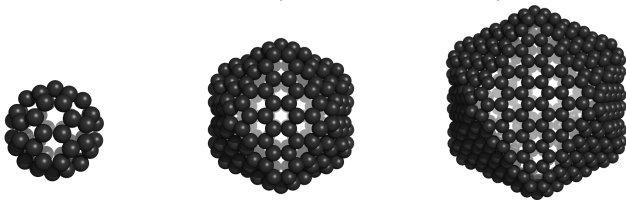
# Use in Mathematical Virology

- Suffice to say **point arrays work very exceedingly well** in practice. Two papers on the mathematical (Coxeter) aspects.
- **Implemented computational problem in Clifford** – some **very interesting mathematics** comes out as well (see later).



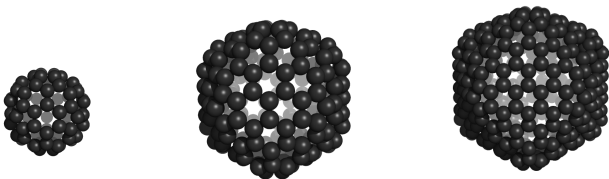
## Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach: **carbon onions** ( $C_{60} - C_{240} - C_{540}$ )



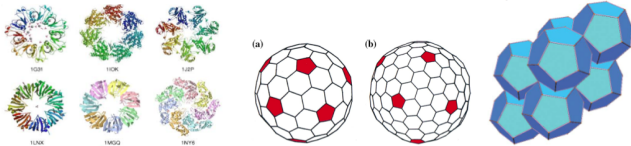
## Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach: **carbon onions** ( $C_{80} - C_{180} - C_{320}$ )



## Applied areas (EPSRC proposal)

- **Viruses:** Extend to large viruses; interesting results on **higher-order translations**
- **Proteins:** Extend affine symmetry to  $2D$  and apply to (**chiral**) proteins
- **Fullerenes:** Extend to larger fullerenes, in particular **chiral carbon onions**
- **Packings:** Novel analytical and numerical approaches to packings of polyhedral solids; with Colapinto (Santa Barbara), Twarock (York) and Thorpe (Phoenix)



## References

- **Novel Kac-Moody-type affine extensions of non-crystallographic Coxeter groups** with Twarock/Bøehm  
J. Phys. A: Math. Theor. 45 285202 (2012)
- **Affine extensions of non-crystallographic Coxeter groups induced by projection** with Twarock/Bøehm  
Journal of Mathematical Physics 54 093508 (2013), [Cover article](#) [September](#)
- **Viruses and Fullerenes – Symmetry as a Common Thread?**  
with Twarock/Wardman/Keef March Cover Acta Crystallographica A (2014)