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Platonic solids generate their four-dimensional analogues

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Root systems – A_2

Root system Φ: set of vectors α such that

1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \; \forall \; \alpha \in \Phi$
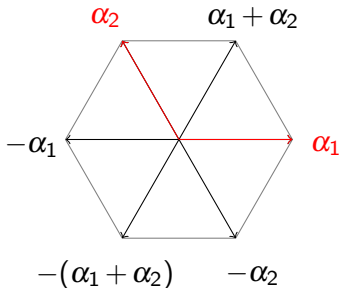
2. $s_\alpha \Phi = \Phi \; \forall \; \alpha \in \Phi$

Simple roots: express every element of Φ via a $\mathbb{Z}$-linear combination (with coefficients of the same sign).

Coxeter groups

A Coxeter group is a group generated by some involutive generators $s_i, s_j \in S$ subject to relations of the form $\boxed{(s_i s_j)^{m_{ij}} = 1}$ with $m_{ij} = m_{ji} \geq 2$ for $i \neq j$.

The finite Coxeter groups have a geometric representation where the involutions are realised as reflections at hyperplanes through the origin in a Euclidean vector space $\mathscr{E}$. In particular, let $(\cdot|\cdot)$ denote the inner product in $\mathscr{E}$, and v, $\alpha \in \mathscr{E}$.

The generator s_α corresponds to the reflection

$$s_\alpha : v \to s_\alpha(v) = v - 2\frac{(v|\alpha)}{(\alpha|\alpha)}\alpha$$

at a hyperplane perpendicular to the root vector α.

The action of the Coxeter group is to permute these root vectors.

Basics of Clifford Algebra I

- Form an algebra using the Geometric Product
  $$ab \equiv a \cdot b + a \wedge b$$ for two vectors

- Extend via linearity and associativity to higher grade elements (multivectors)

- For an n-dimensional space generated by n orthogonal unit vectors e_i have 2^n elements

- Then $$e_i e_j = e_i \wedge e_j = -e_j e_i$$ so anticommute (Grassmann variables, exterior algebra)

- Unlike the inner and outer products separately, this product is invertible

Basics of Clifford Algebra II

- These are known to have matrix representations over the normed division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ $\Rightarrow$ Classification of Clifford algebras

- E.g. Pauli algebra in 3D (likewise for Dirac algebra in 4D) is

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1 e_2 e_3\}}_{1 \text{ trivector}}$$

- These have the well-known matrix representations in terms of σ- and γ-matrices

- Working with these is not necessarily the most insightful thing to do, so here stress approach to work directly with the algebra

Reflections

- Clifford algebra is very efficient at performing reflections
- Consider reflecting the vector a in a hypersurface with unit normal n:

$$a' = a_\perp - a_\parallel = a - 2a_\parallel = a - 2(a \cdot n)n$$

- c.f. fundamental Weyl reflection $s_i : v \to s_i(v) = v - 2\frac{(v|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i$
- But in Clifford algebra have $n \cdot a = \frac{1}{2}(na + an)$ so reassembles into (note doubly covered by n and $-n$) sandwiching

$$\boxed{a' = -nan}$$

- So both Coxeter and Clifford frameworks are ideally suited to describing reflections – combine the two

Reflections and Rotations

- Generate a rotation when compounding two reflections wrt n then m (Cartan-Dieudonné theorem):

$$a'' = mnanm \equiv Ra\tilde{R}$$

where $R = mn$ is called a spinor and a tilde denotes reversal of the order of the constituent vectors ($R\tilde{R} = 1$)

- All multivectors transform covariantly e.g.

$$MN \rightarrow (RM\tilde{R})(RN\tilde{R}) = RM\tilde{R}RN\tilde{R} = R(MN)\tilde{R}$$

so transform double-sidedly

- Spinors form a group, which gives a representation of the Spin group $Spin(n)$ – they transform single-sidedly (obvious it's a double (universal) cover)

Artin's Theorem and orthogonal transformations

- **Artin**: every isometry is at most d reflections
- Since have a **double cover** of reflections (n and $-n$) we have a **double cover** of $O(p, q)$: $\text{Pin}(p, q)$

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1$$

- **Pinors** = products of vectors $n_1 n_2 \dots n_k$ encode orthogonal transformations via 'sandwiching'
- **Cartan-Dieudonné**: rotations are an **even** number of reflections: $\text{Spin}(p, q)$ doubly covers $SO(p, q)$

3D Platonic Solids

- There are 5 Platonic solids
- Tetrahedron (self-dual) (A_3)
- Dual pair octahedron and cube (B_3)
- Dual pair icoshahedron and dodecahedron (H_3)
- Only the octahedron is a root system (actually for (A_1^3))

Clifford and Coxeter: Platonic Solids

Platonic Solid	Group	root system
Tetrahedron	A_3	Cuboctahedron
	A_1^3	Octahedron
Octahedron	B_3	Cuboctahedron
Cube		+Octahedron
Icosahedron	H_3	Icosidodecahedron
Dodecahedron		

- Platonic Solids have been known for millennia
- Described by Coxeter groups

4D 'Platonic Solids'

- In 4D, there are 6 analogues of the Platonic Solids:
- 5-cell (self-dual) (A_4)
- 24-cell (self-dual) (D_4) – a 24-cell and its dual together are the F_4 root system
- Dual pair 16-cell and 8-cell (B_4)
- Dual pair 600-cell and 120-cell (H_4)
- These are 4D analogues of the Platonic Solids: regular convex 4-polytopes
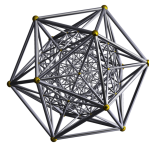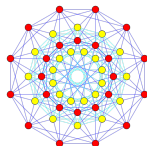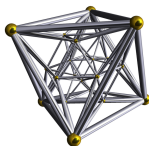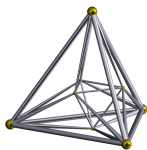
4D 'Platonic Solids'

- 24-cell, 16-cell and 600-cell are all root systems, as is the related F_4 root system

- 8-cell and 120-cell are dual to a root system, so in 4D out of 6 Platonic Solids only the 5-cell (corresponding to A_n family) is not related to a root system!

- The 4D Platonic solids are not normally thought to be related to the 3D ones except for the boundary cells

- They have very unusual automorphism groups

- Some partial case-by-case algebraic results in terms of quaternions – here we show a uniform construction offering geometric understanding

Mysterious Symmetries of 4D Polytopes

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Spinorial symmetries

rank 4	$|\Phi|$	Symmetry
D_4 24-cell	24	$2 \cdot 24^2 = 576$
F_4 lattice	48	$48^2 = 2304$
H_4 600-cell	120	$120^2 = 14400$
A_1^4 16-cell	8	$3! \cdot 8^2 = 384$
$A_2 \oplus A_2$ prism	12	$12^2 = 144$
$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

</div>

Similar for Grand Antiprism (H_4 without $H_2 \oplus H_2$) and Snub
24-cell ($2I$ without $2T$).

A new connection

A_1^3 A_1^4

A_3 D_4

B_3 F_4

H_3 H_4

- Platonic Solids have been known for millennia; described by Coxeter groups
- Concatenating reflections gives Clifford spinors (binary polyhedral groups)
- These induce 4D root systems
  $$\psi = a_0 + a_i I e_i \Rightarrow \psi \tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$
- 4D analogues of the Platonic Solids and give rise to 4D Coxeter groups
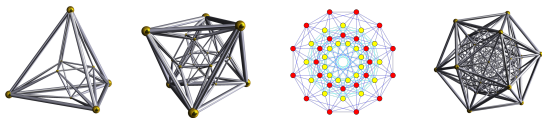
Overview

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Induction Theorem – root systems

- Theorem: 3D spinor groups give root systems.
- Proof: 1. R and $-R$ are in a spinor group by construction, 2. closure under reflections is guaranteed by the closure property of the spinor group
- Induction Theorem: Every rank-3 root system induces a rank-4 root system (and thereby Coxeter groups)
- Counterexample: not every rank-4 root system is induced in this way

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Induction Theorem – automorphism

- So induced 4D polytopes are actually root systems.

- Clear why the number of roots $|\Phi|$ is equal to $|G|$, the order of the spinor group

- Spinor group is trivially closed under conjugation, left and right multiplication. Results in non-trivial symmetries when viewed as a polytope/root system.

- Now explains symmetry of the polytopes/root system and thus the order of the rank-4 Coxeter group

- Theorem: The automorphism group of the induced root system contains two factors of the respective spinor group acting from the left and the right.

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Recap: Clifford algebra and reflections & rotations

- Clifford algebra is very efficient at performing reflections via sandwiching

$$a' = -nan$$

- Generate a rotation when compounding two reflections wrt n then m (Cartan-Dieudonné theorem):

$$a'' = mnanm \equiv Ra\tilde{R}$$
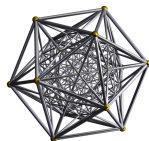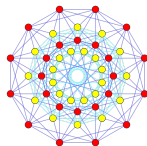
where $R = mn$ is called a spinor and a tilde denotes reversal of the order of the constituent vectors ($R\tilde{R} = 1$)

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Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the Platonic Solids:

- The $6/12/18/30$ reflections in $A_1 \times A_1 \times A_1/A_3/B_3/H_3$ generate $8/24/48/120$ spinors.

- E.g. $\pm e_1$, $\pm e_2$, $\pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$

- The discrete spinor group is isomorphic to the quaternion group Q / binary tetrahedral group $2T$ / binary octahedral group $2O$/ binary icosahedral group $2I$).

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Spinors and Polytopes

- The space of Cl(3)-spinors and quaternions have a 4D Euclidean signature: $\psi = a_0 + a_i l e_i \Rightarrow \psi \tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- Can reinterpret spinors in $\mathbb{R}^3$ as vectors in $\mathbb{R}^4$
- Then the spinors constitute the vertices of the 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell
- These are 4D analogues of the Platonic Solids: regular convex 4-polytopes

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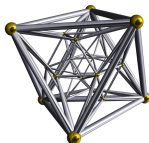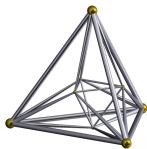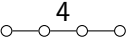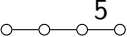
Exceptional Root Systems

- The 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell are in fact the root systems of $A_1 \times A_1 \times A_1 \times A_1$, D_4, F_4 and H_4

- Exceptional phenomena: D_4 (triality, important in string theory), F_4 (largest lattice symmetry in 4D), H_4 (largest non-crystallographic symmetry)

- Exceptional D_4 and F_4 arise from series A_3 and B_3

- In fact, as we have seen one can strengthen this statement on inducing polytopes to a statement on inducing root systems

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Root systems in three and four dimensions

The spinors generated from the reflections contained in the respective rank-3 Coxeter group via the geometric product are realisations of the binary polyhedral groups Q, $2T$, $2O$ and $2I$, which were known to generate (mostly exceptional) rank-4 groups, but not known why, and why the 'mysterious symmetries'.

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$	○ ○ ○	Q	$A_1 \times A_1 \times A_1 \times A_1$	○ ○ ○ ○
A_3	○—○—○	$2T$	D_4	○—○⟨○ ○
B_3	○—○—⁴○	$2O$	F_4	○—○—⁴○—○
H_3	○—○—⁵○	$2I$	H_4	○—○—○—⁵○

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General Case of Induction

Only remaining case is what happens for $A_1 \oplus I_2(n)$ - this gives a doubling $I_2(n) \oplus I_2(n)$

rank 3	rank 4
A_3	D_4
B_3	F_4
H_3	H_4
A_1^3	A_1^4
$A_1 \oplus A_2$	$A_2 \oplus A_2$
$A_1 \oplus H_2$	$H_2 \oplus H_2$
$A_1 \oplus I_2(n)$	$I_2(n) \oplus I_2(n)$

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Automorphism Groups

- So induced 4D polytopes are actually root systems via the binary polyhedral groups.

- Clear why the number of roots $|\Phi|$ is equal to $|G|$, the order of the spinor group.

- Spinor group is trivially closed under conjugation, left and right multiplication. Results in non-trivial symmetries when viewed as a polytope/root system.

- Now explains symmetry of the polytopes/root system and thus the order of the rank-4 Coxeter group

- Theorem: The automorphism group of the induced root system contains two factors of the respective spinor group acting from the left and the right.

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Spinorial Symmetries of 4D Polytopes

<p align="center">Spinorial symmetries</p>

rank 3	$|\Phi|$	$|W|$	rank 4	$|\Phi|$	Symmetry
A_3	12	24	D_4 24-cell	24	$2 \cdot 24^2 = 576$
B_3	18	48	F_4 lattice	48	$48^2 = 2304$
H_3	30	120	H_4 600-cell	120	$120^2 = 14400$
A_1^3	6	8	A_1^4 16-cell	8	$3! \cdot 8^2 = 384$
$A_1 \oplus A_2$	8	12	$A_2 \oplus A_2$ prism	12	$12^2 = 144$
$A_1 \oplus H_2$	12	20	$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$A_1 \oplus I_2(n)$	$n+2$	$2n$	$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

Similar for Grand Antiprism (H_4 without $H_2 \oplus H_2$) and Snub 24-cell ($2I$ without $2T$). Additional factors in the automorphism group come from 3D Dynkin diagram symmetries!

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Some non-Platonic examples of spinorial symmetries

- Grand Antiprism: the 100 vertices achieved by subtracting 20 vertices of $H_2 \oplus H_2$ from the 120 vertices of the H_4 root system 600-cell – two separate orbits of $H_2 \oplus H_2$

- This is a semi-regular polytope with automorphism symmetry $\mathrm{Aut}(H_2 \oplus H_2)$ of order $400 = 20^2$

- Think of the $H_2 \oplus H_2$ as coming from the doubling procedure? (Likewise for $\mathrm{Aut}(A_2 \oplus A_2)$ subgroup)

- Snub 24-cell: $2T$ is a subgroup of $2I$ so subtracting the 24 corresponding vertices of the 24-cell from the 600-cell, one gets a semiregular polytope with 96 vertices and automorphism group $2T \times 2T$ of order $576 = 24^2$.

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Sub root systems

- The above spinor groups had spinor multiplication as the group operation

- But also closed under twisted conjugation – corresponds to closure under reflections (root system property)

- If we take twisted conjugation as the group operation instead, we can have various subgroups

- These are the remaining 4D root systems e.g. A_4 or B_4

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Arnold's Trinities

Arnold's observation that many areas of real mathematics can be complexified and quaternionified resulting in theories with a similar structure.

- The fundamental trinity is thus $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The projective spaces $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The spheres $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The Möbius/Hopf bundles $(S^1 \to S^1, S^4 \to S^2, S^7 \to S^4)$
- The Lie Algebras (E_6, E_7, E_8)
- The symmetries of the Platonic Solids (A_3, B_3, H_3)
- The 4D groups (D_4, F_4, H_4)
- New connections via my Clifford spinor construction (see McKay correspondence)

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Platonic Trinities

- Arnold's connection between (A_3, B_3, H_3) and (D_4, F_4, H_4) is very convoluted and involves numerous other trinities at intermediate steps:

- Decomposition of the projective plane into Weyl chambers and Springer cones

- The number of Weyl chambers in each segment is $24 = 2(1+3+3+5), 48 = 2(1+5+7+11), 120 = 2(1+11+19+29)$

- Notice this miraculously matches the quasihomogeneous weights $((2,4,4,6),(2,6,8,12),(2,12,20,30))$ of the Coxeter groups (D_4, F_4, H_4)

- Believe the Clifford connection is more direct

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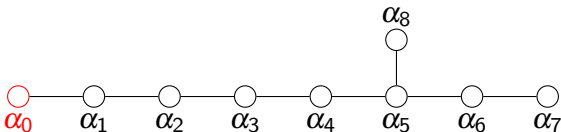**Combining Coxeter and Clifford**
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A unified framework for polyhedral groups

Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm\tilde{A}xA$
$\text{Spin}(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$\text{Pin}(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the chiral icosahedral group has 60 elements, encoded in Clifford by 120 spinors, which form the binary icosahedral group
- together with the inversion/pseudoscalar I this gives 60 rotations and 60 rotoinversions, i.e. the full icosahedral group H_3 in 120 elements (with 240 pinors)
- all three are interesting groups, e.g. in neutrino and flavour physics for family symmetry model building

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Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate conjugacy classes etc of pinors in Clifford algebra
- Chiral (binary) polyhedral groups have irreps
- tetrahedral (12/24): 1, $1'$, $1''$, 2_s, $2'_s$, $2''_s$, 3
- octahedral (24/48): 1, $1'$, 2, 2_s, $2'_s$, 3, $3'$, 4_s
- icosahedral (60/120): 1, 2_s, $2'_s$, 3, $\bar{3}$, 4, 4_s, 5, 6_s
- Binary groups are discrete subgroups of $SU(2)$ and all thus have a 2_s spinor irrep
- Connection with the McKay correspondence!

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Affine extensions – $E_8^=$

$$-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$$

AKA E_8^+ and along with E_8^{++} and E_8^{+++} thought to be the underlying symmetry of String and M-theory
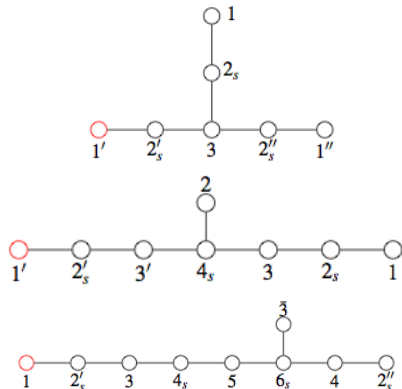
Also interesting from a pure mathematics point of view: E_8 lattice, McKay correspondence and Monstrous Moonshine.

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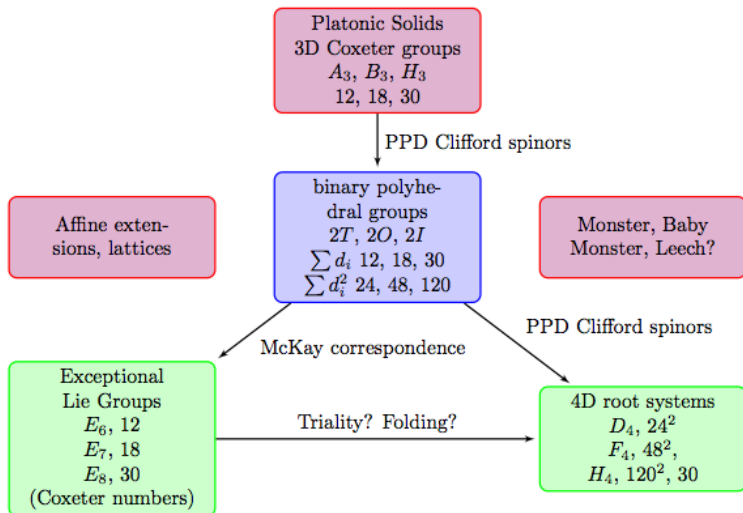
The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the
cyclic and dicyclic groups are in correspondence with A_n and D_n,
e.g. the quaternion group Q and D_4^+. So McKay correspondence
not just a trinity but ADE-classification. We also have $I_2(n)$ on top
of the trinity (A_3, B_3, H_3)

rank-3 group	diagram	binary	rank-4 group	diagram	Lie algebra	diagram
$A_1 \times A_1 \times A_1$	○ ○ ○	Q	$A_1 \times A_1 \times A_1 \times A_1$	○ ○ ○ ○	D_4^+	
A_3	○—○—○	$2T$	D_4	○—○—○	E_6^+	
B_3	○—○—○ (4)	$2O$	F_4	○—○—○—○ (4)	E_7^+	
H_3	○—○—○ (5)	$2I$	H_4	○—○—○ (5)	E_8^+	

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4D geometry is surprisingly important for HEP

- 4D root systems are surprisingly relevant to HEP
- A_4 is $SU(5)$ and comes up in Grand Unification
- D_4 is $SO(8)$ and is the little group of String theory
- In particular, its triality symmetry is crucial for showing the equivalence of RNS and GS strings
- B_4 is $SO(9)$ and is the little group of M-Theory
- F_4 is the largest crystallographic symmetry in 4D and H_4 is the largest non-crystallographic group
- The above are subgroups of the latter two
- Spinorial nature of the root systems could have surprising consequences for HEP

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References (single-author)

- Clifford algebra unveils a surprising geometric significance of quaternionic root systems of Coxeter groups
  Advances in Applied Clifford Algebras, June 2013, Volume 23, Issue 2, pp 301-321

- A Clifford algebraic framework for Coxeter group theoretic computations (Conference Prize at AGACSE 2012)
  Advances in Applied Clifford Algebras 24 (1). pp. 89-108 (2014)

- Rank-3 root systems induce root systems of rank 4 via a new Clifford spinor construction arXiv:1207.7339 (2012)

- Platonic Solids generate their 4-dimensional analogues
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Conclusions

- Novel connection between geometry of 3D and 4D
- In fact, 3D seems more fundamental – contrary to the usual perspective of 3D subgroups of 4D groups
- Spinorial symmetries
- Clear why spinor group gives a root system and why two factors of the same group reappear in the automorphism group
- Novel spinorial perspective on 4D geometry
- Accidentalness of the spinor construction and exceptional 4D phenomena
- Connection with Arnold's trinities, the McKay correspondence and Monstrous Moonshine

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Thank you!

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Motivation: Viruses

- Geometry of polyhedra described by Coxeter groups
- Viruses have to be 'economical' with their genes
- Encode structure modulo symmetry
- Largest discrete symmetry of space is the icosahedral group
- Many other 'maximally symmetric' objects in nature are also icosahedral: Fullerenes & Quasicrystals
- But: viruses are not just polyhedral – they have radial structure. Affine extensions give translations
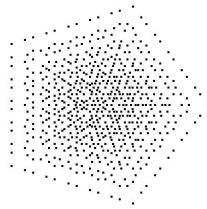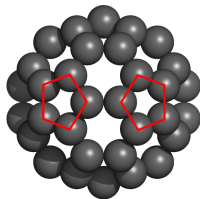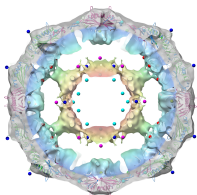
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Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon

G

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Unit translation along a vertex of a unit pentagon

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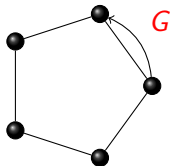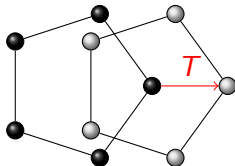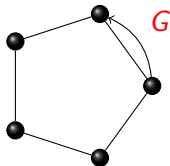
Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon
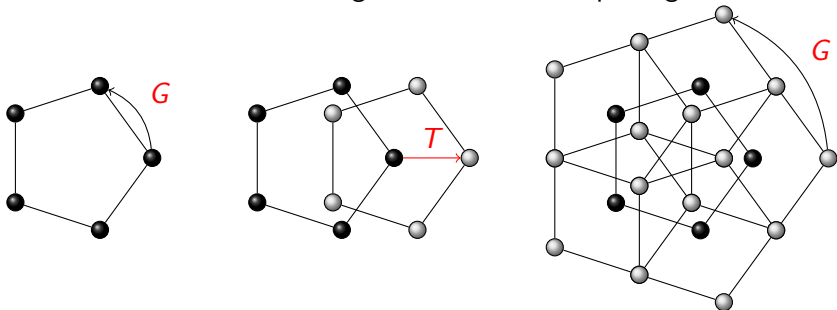
A random translation would give 5 secondary pentagons, i.e. 25 points. Here we have degeneracies due to 'coinciding points'.

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Affine extensions of non-crystallographic root systems

Translation of length $\tau = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ (golden ratio)
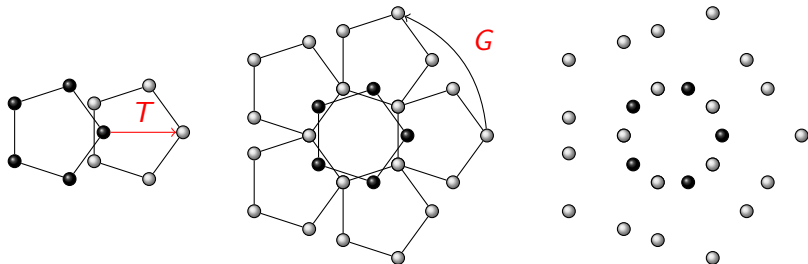
Looks like a virus or carbon onion

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Extend icosahedral group with distinguished translations

- Radial layers are simultaneously constrained by affine symmetry
- Works very well in practice: finite library of blueprints
- Select blueprint from the outer shape (capsid)
- Can predict inner structure (nucleic acid distribution) of the virus from the point array
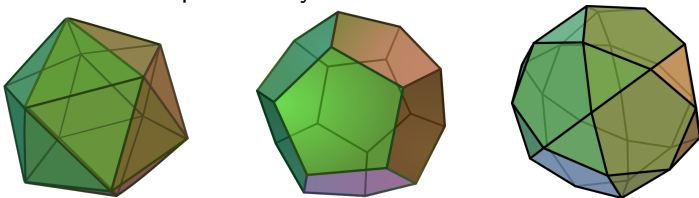
Affine extensions of the icosahedral group (giving translations) and their classification.

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Use in Mathematical Virology

- Suffice to say point arrays work very exceedingly well in practice. Two papers on the mathematical (Coxeter) aspects.
- Implemented computational problem in Clifford – some very interesting mathematics comes out as well (see later).
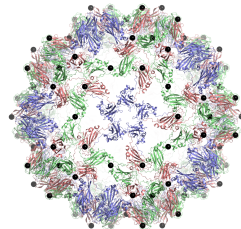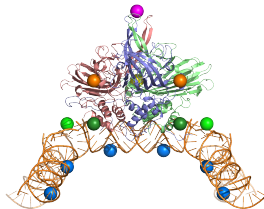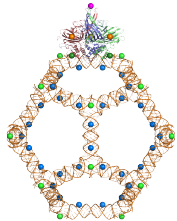
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Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{60} - C_{240} - C_{540}$)
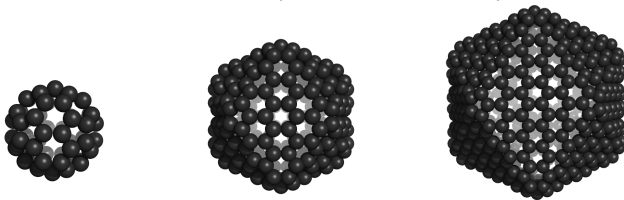
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Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{80} - C_{180} - C_{320}$)

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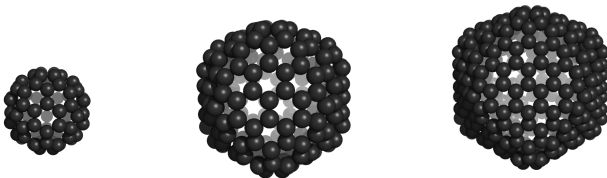
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- Affine extensions of non-crystallographic Coxeter groups induced by projection with Twarock/Bœhm Journal of Mathematical Physics 54 093508 (2013), Cover article September

- Viruses and Fullerenes – Symmetry as a Common Thread? with Twarock/Wardman/Keef March Cover Acta Crystallographica A 70 (2). pp. 162-167 (2014), and Nature Physics Research Highlight

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Applications of affine extensions of non-crystallographic root systems

There are interesting applications to quasicrystals, viruses or carbon onions, but here concentrate on the mathematical aspects

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Quaternions and Clifford Algebra

- The unit spinors $\{1; Ie_1; Ie_2; Ie_3\}$ of Cl(3) are isomorphic to the quaternion algebra $\mathbb{H}$ (up to sign)
- The 3D Hodge dual of a vector is a pure bivector which corresponds to a pure quaternion, and their products are identical (up to sign)
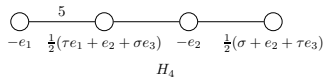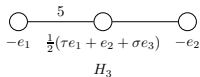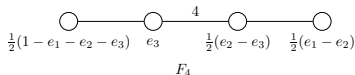
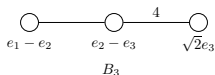Introduction
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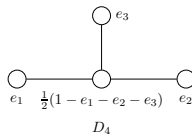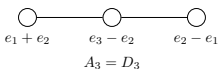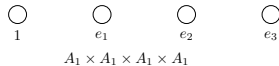Automorphism Groups
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Discrete Quaternion groups

- The 8 quaternions of the form $(\pm 1, 0, 0, 0)$ and permutations are called the Lipschitz units, and form a realisation of the quaternion group in 8 elements.

- The 8 Lipschitz units together with $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ are called the Hurwitz units, and realise the binary tetrahedral group of order 24. Together with the 24 'dual' quaternions of the form $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$, they form a group isomorphic to the binary octahedral group of order 48.

- The 24 Hurwitz units together with the 96 unit quaternions of the form $(0, \pm \tau, \pm 1, \pm \sigma)$ and even permutations, are called the Icosians. The icosian group is isomorphic to the binary icosahedral group with 120 elements.

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Quaternionic representations of 3D and 4D Coxeter groups

- Groups E_8, D_4, F_4 and H_4 have representations in terms of quaternions
- Extensively used in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why
- e.g. H_4 consists of 120 elements of the form $(\pm 1, 0, 0, 0)$, $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and $(0, \pm \tau, \pm 1, \pm \sigma)$
- Seen as remarkable that the subset of the 30 pure quaternions is a realisation of H_3 (a sub-root system)
- Similarly, A_3, B_3, $A_1 \times A_1 \times A_1$ have representations in terms of pure quaternions
- Will see there is a much simpler geometric explanation

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Quaternionic representations used in the literature

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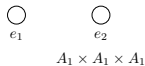
Demystifying Quaternionic Representations

- 3D: Pure quaternions = Hodge dualised (pseudoscalar) root vectors
- In fact, they are the simple roots of the Coxeter groups
- 4D: Quaternions = disguised spinors – but those of the 3D Coxeter group i.e. the binary polyhedral groups!
- This relation between 3D and 4D via the geometric product does not seem to be known
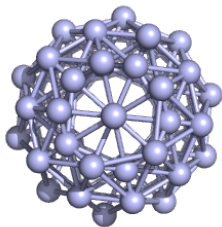- Quaternion multiplication = ordinary Clifford reflections and rotations

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Demystifying Quaternionic Representations

- Pure quaternion subset of 4D groups only gives 3D group if the 3D group contains the inversion/pseudoscalar I

- e.g. does not work for the tetrahedral group A_3, but $A_3 \to D_4$ induction still works, with the central node essentially 'spinorial'

- In fact, it goes the other way around: the 3D groups induce the 4D groups via spinors

- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as $R_1 = \alpha_1 \alpha_2$ and $R_2 = \alpha_2 \alpha_3$

- Can see these are 'spinor generators' and how they don't really contain any more information/roots than the rank-3 groups alone

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Quaternions vs Clifford versors

- Sandwiching is often seen as particularly nice feature of the quaternions giving rotations

- This is actually a general feature of Clifford algebras/versors in any dimension; the isomorphism to the quaternions is accidental to 3D

- However, the root system construction does not necessarily generalise

- 2D generalisation merely gives that $I_2(n)$ is self-dual

- Octonionic generalisation just induces two copies of the above 4D root systems, e.g. $A_3 \rightarrow D_4 \oplus D_4$

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A new connection