A new take on polyhedral things

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Applied Algebra at Oxford – June 3, 2016

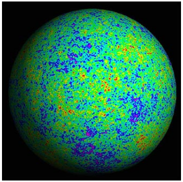
Eclectic interests – but the general theme is Geometry & Symmetry and their Applications

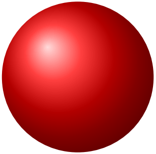
- Worked on a few different things: HEP – strings, particles and cosmology, pure maths and mathematical biology and Clifford algebras and mathematical physics
- Unifying themes of symmetry and geometry (euclidean, conformal, hyperbolic, spherical)
- Continuous Lie groups, e.g. for modeling cosmological spacetimes (Bianchi models), gauge symmetries, compactifications &c
- Discrete Coxeter groups and Kac-Moody algebras describe gravitational singularities/hidden symmetries in HEP theory, viruses, fullerenes, &c

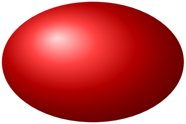
What's new?

- In HEP, mostly come from Lie groups, then Lie algebras, then their Weyl groups and root systems
- This only gives the crystallographic Coxeter groups
- Do the non-crystallographic Coxeter groups have something interesting to offer? In particular, affine extensions?
- Interesting connections between the geometries of different dimensions: Relation between crystallographic and non-crystallographic (E_8 and H_4) and my spinor construction (3D & 4D (D_4, F_4, H_4), 8D (E_8))
- Both could have interesting consequences for HEP (4D groups and E_8 feature heavily) and other applications (viruses, quasicrystals, proteins, fullerenes...)

Singularities in cosmology/general relativity

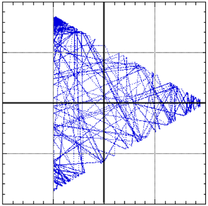






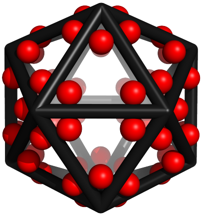
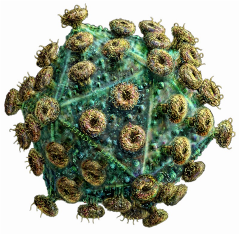
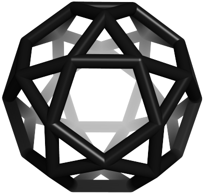
- In PhD, was looking at non-singular models of the universe (topologically S^3) using conformal geometry and Clifford algebra techniques
- Rather difficult to arrange – need very special conditions
- Generic case: there are singularities (Hawking and Penrose)
- Analytic structure/approach to singularity described by hyperbolic Coxeter groups

PhD: Theoretical Cosmology



- Analytic structure/approach to singularity described by hyperbolic Coxeter groups
- Actually holds for large class of gravitational theories in various dimensions (general relativity, supergravity, string derived models)
- Damour-Henneaux-Nicolai conjecture: These are the Weyl groups of some underlying Lorentzian Kac-Moody algebra symmetry of the gravitational theory

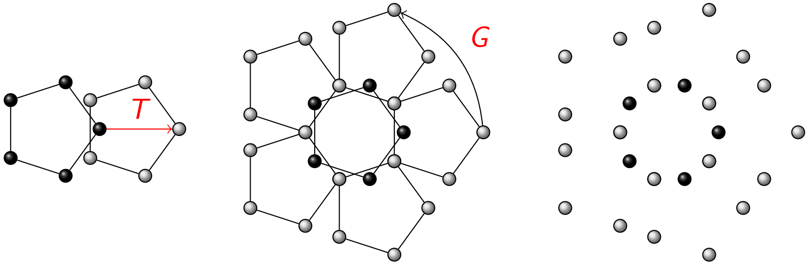
Icosahedral Viruses

- Rotational icosahedral group is $I = A_5$ of order 60
- Full icosahedral group is H_3 of order 120 (including reflections/inversion); generated by the root system icosidodecahedron

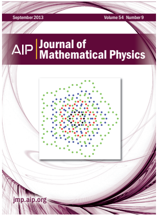
Affine extensions of non-crystallographic Coxeter groups?

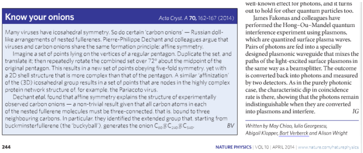
Translation of length $\tau = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$ (golden ratio)



Cartoon version of a virus or carbon onion. Would there be an evolutionary benefit to have more than just compact symmetry? The problem has an intrinsic length scale.

- 2D and 3D point arrays for applications to viruses, fullerenes, quasicrystals etc
- Two complementary ways to construct these





Know your onions *Acta Cryst. A* **70,** 162–167 (2014)

Many viruses have icosahedral symmetry. So do certain 'carbon onions' — Russian doll-like arrangements of nested fullerenes. Pierre-Philippe Dechant and colleagues argue that viruses and carbon onions share the same formation principle: affine symmetry.

Imagine a set of points lying on the vertices of a regular pentagon. Duplicate the set, and translate it; then repeatedly rotate the combined set over 72° about the midpoint of the original pentagon. This results in a new set of points obeying five-fold symmetry, yet with a 2D shell structure that is more complex than that of the pentagon. A similar 'affinization' of the (3D) icosahedral group results in a set of points that are nodes in the highly complex protein network structure of, for example, the Pariacoto virus.

Dechant *et al.* found that affine symmetry explains the structure of experimentally observed carbon onions — a non-trivial result given that all carbon atoms in each of the nested fullerene molecules must be three-connected, that is, bound to three neighbouring carbons. In particular, they identified the extended group that, starting from buckminsterfullerene (the 'buckyball'), generates the onion $C_{60}@C_{240}@C_{540}$. *BV*

well-known effect for photons, and it turns out to hold for other quantum particles too.

James Fakonas and colleagues have performed the Hong–Ou–Mandel quantum interference experiment using plasmons, which are quantized surface plasma waves. Pairs of photons are fed into a specially designed plasmonic waveguide that mixes the paths of the light-excited surface plasmons in the same way as a beamsplitter. The outcome is converted back into photons and measured by two detectors. As in the purely photonic case, the characteristic dip in coincidence rate is there, showing that the photons remain indistinguishable when they are converted into plasmons and interfere. *IG*

Written by May Chiao, Iulia Georgescu, Abigail Klopper, Bart Verberck and Alison Wright

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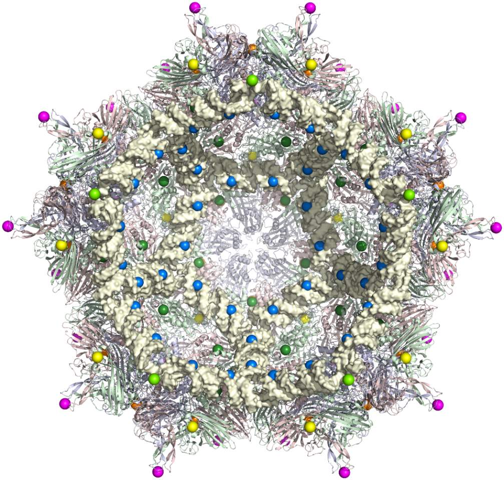
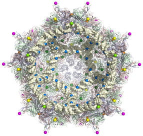
Use in Mathematical Virology

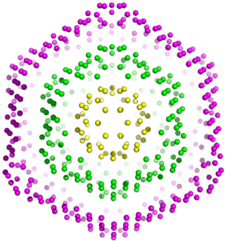


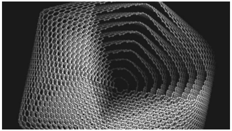
New insight into RNA virus assembly

- There are specific interactions between RNA and coat protein (CP) given by symmetry axes
- Essential for assembly as only this RNA-CP interaction turns CP into right geometric shape for capsid formation
- The RNA forms a Hamiltonian cycle visiting each CP once – dictated by symmetry
- A patent for a new antiviral strategy (Reidun Twarock)

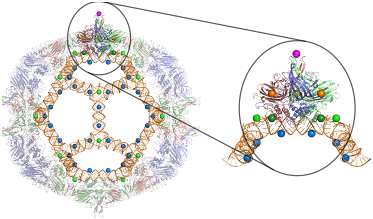
- Get nested arrangements like Russian dolls: carbon onions (e.g. June: Nature 510, 250253)
- Potential to extend to other known carbon onions with different start configuration, chirality etc

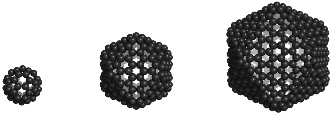




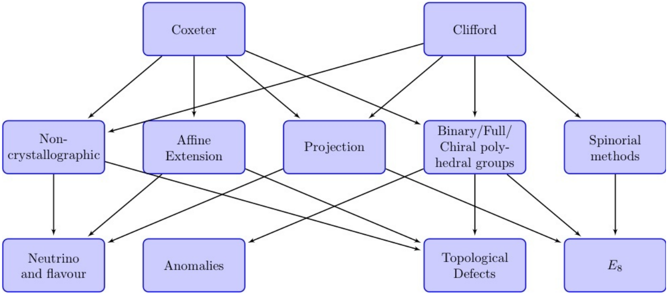
Two major areas for Affine extensions of non-crystallographic Coxeter groups

- Non-compact symmetry that relates different structural features in the same polyhedral object
- Novel symmetry principle in Nature, shown that it seems to apply to at least fullerenes and viruses

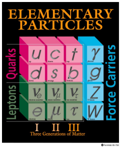


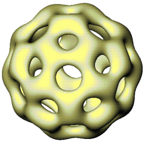


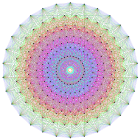
Applications of these group structures in particle physics



University of Durham

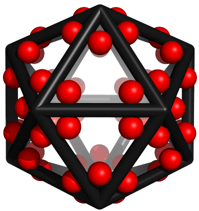


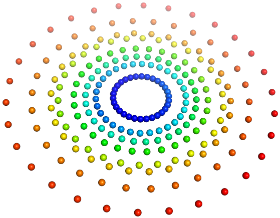


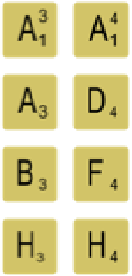


Purer Aspects of the Geometry

- Coxeter plane geometry, quaternionic representations, modular group etc
- Each 3D root system induces a 4D root system
- H_3 (icosahedral symmetry) induces the E_8 root system
- Clifford algebra is a very natural framework for root systems and reflection groups







Clifford Algebra and orthogonal transformations

- Form an algebra using the Geometric Product for two vectors

$$\boxed{ab \equiv a \cdot b + a \wedge b}$$

- Inner product is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting a in b is given by $\boxed{a' = a - 2(a \cdot b)b = -bab}$ (b and $-b$ doubly cover the same reflection)
- Via Cartan-Dieudonné theorem any orthogonal (/conformal/modular) transformation can be written as successive reflections

$$\boxed{x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1} = \pm A x \tilde{A}$$

Clifford Algebra of 3D

- E.g. Pauli algebra in 3D (likewise for Dirac algebra in 4D) is

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

- We can multiply together root vectors in this algebra $\alpha_i\alpha_j\ldots$
- A general element has 8 components, even products (rotations/spinors) have four components:

$$\boxed{R = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2 \Rightarrow R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

- So behaves as a 4D Euclidean object – inner product

$$\boxed{(R_1, R_2) = \frac{1}{2}(R_2\tilde{R}_1 + R_1\tilde{R}_2)}$$

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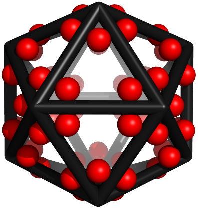
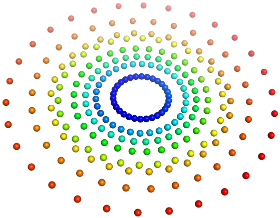
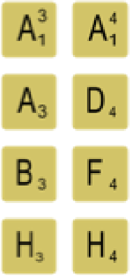

Thank you!