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# A new take on polyhedral things

Pierre-Philippe Dechant

Department of Mathematics, University of York

Applied Algebra at Oxford – June 3, 2016

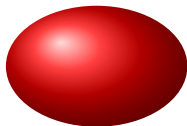
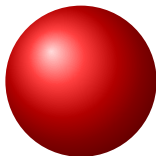
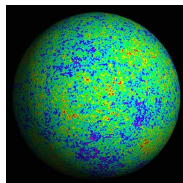
# Eclectic interests – but the general theme is Geometry & Symmetry and their Applications

- Worked on a few different things: HEP – strings, particles and cosmology, pure maths and mathematical biology and Clifford algebras and mathematical physics
- Unifying themes of symmetry and geometry (euclidean, conformal, hyperbolic, spherical)
- Continuous Lie groups, e.g. for modeling cosmological spacetimes (Bianchi models), gauge symmetries, compactifications &c
- Discrete Coxeter groups and Kac-Moody algebras describe gravitational singularities/hidden symmetries in HEP theory, viruses, fullerenes, &c

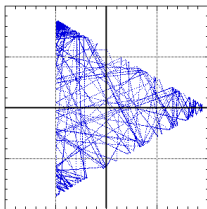
# What's new?

- In HEP, mostly come from **Lie groups**, then **Lie algebras**, then their **Weyl groups** and **root systems**
- This only gives the **crystallographic Coxeter** groups
- Do the **non-crystallographic** Coxeter groups have something interesting to offer? In particular, **affine extensions**?
- Interesting connections between the **geometries of different dimensions**: Relation between **crystallographic and non-crystallographic** ( $E_8$  and  $H_4$ ) and my **spinor construction** (3D & 4D ( $D_4, F_4, H_4$ ), 8D ( $E_8$ ))
- Both could have **interesting consequences for HEP** (4D groups and  $E_8$  feature heavily) and other applications (**viruses, quasicrystals, proteins, fullerenes...**)

# Singularities in cosmology/general relativity

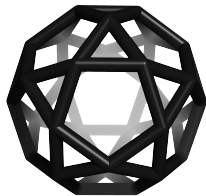
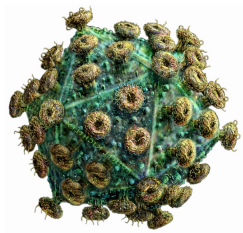
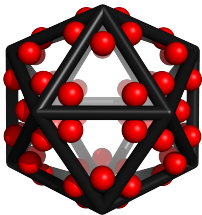


- In PhD, was looking at **non-singular models** of the universe (topologically  $S^3$ ) using **conformal geometry** and **Clifford algebra techniques**
- Rather difficult to arrange – need **very special** conditions
- **Generic** case: there are singularities (**Hawking and Penrose**)
- **Analytic structure**/approach to singularity described by **hyperbolic Coxeter groups**



- **Analytic structure**/approach to singularity described by **hyperbolic Coxeter groups**
- Actually holds for **large class of gravitational theories in various dimensions** (general relativity, supergravity, string derived models)
- **Damour-Henneaux-Nicolai conjecture**: These are the Weyl groups of some underlying **Lorentzian Kac-Moody algebra symmetry** of the gravitational theory

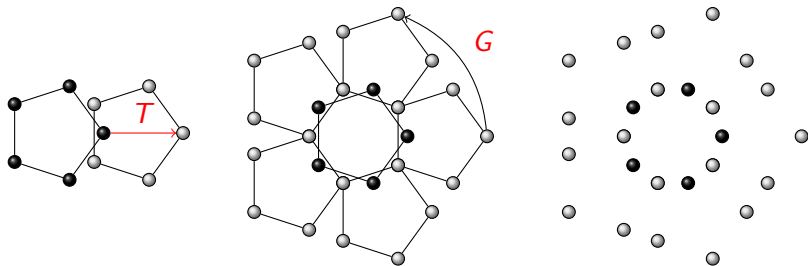
# Icosahedral Viruses



- **Rotational** icosahedral group is  $I = A_5$  of order **60**
- **Full** icosahedral group is  $H_3$  of order **120** (including reflections/inversion); generated by the root system icosidodecahedron

# Affine extensions of non-crystallographic Coxeter groups?

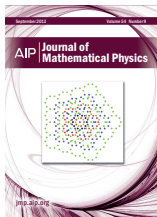
Translation of length  $\tau = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$  (golden ratio)



Cartoon version of a **virus** or **carbon onion**. Would there be an **evolutionary benefit** to have more than just compact symmetry?  
The problem has an **intrinsic length scale**.

# Affine extensions of non-crystallographic Coxeter groups

- 2D and 3D **point arrays** for applications to viruses, fullerenes, quasicrystals etc
- **Two complementary ways** to construct these



## Know your onions

Adv. Phys. 47(1), 103-167 (2014)

Many viruses have icosahedral symmetry. So do certain 'carbon onions' — Russian doll-like arrangements of nested fullerenes. Pierre-Philippe Dechant and colleagues argue that viruses and carbon onions share the same formation principle: affine symmetry. Imagine a set of points lying on the vertices of a regular pentagon. Duplicate the set, and translate it, then repeatedly rotate the combined set over  $72^\circ$  about the midpoint of the original pentagon. This results in a new set of points obeying five-fold symmetry, set within a 2D shell structure that is more complex than that of the pentagon. A similar application of the (3D) icosahedral group results in a set of points that are nodes in the highly complex protein network structure of, for example, the Porcine virus.

Dechant et al. found that affine symmetry explains the structure of experimentally observed carbon onions — a non-trivial result given that all carbon atoms in each of the nested fullerene molecules must be three-connected, that is, bound to three neighbouring carbons. In particular, they identified the extended group that, starting from buckminsterfullerene (the 'buckyball'), generates the onion  $C_{60} @ C_{60} @ C_{60}$ . **BV**

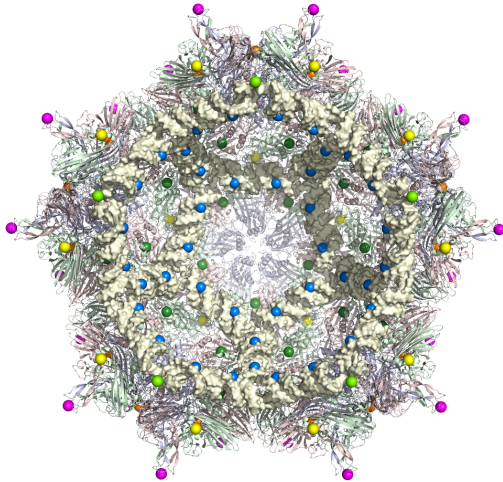
well-known effect for photons, and it turns out to hold for other quantum particles too. James Follis and colleagues have performed the Hong-Ou-Mandel quantum interference experiment using plasmons, which are quantised surface plasmon waves. Pairs of photons are fed into a specially designed plasmonic waveguide that mixes the paths of the light excited surface plasmons in the same way as a beam splitter. The outcome is converted back into photons and measured by two detectors. As in the purely photonic case, the characteristic dip in coincidence rate is there, showing that the photons remain indistinguishable when they are converted into plasmons and interfere. **IG**

Written by May Chiu, Milo Gregoric, Abigail Knappe, Bart Verbruggen and Alison Wright

NATURE PHYSICS | VOL 10 | APRIL 2014 | www.nature.com/naturephysics

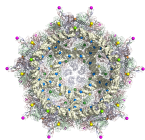
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# Use in Mathematical Virology



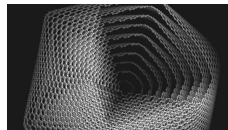
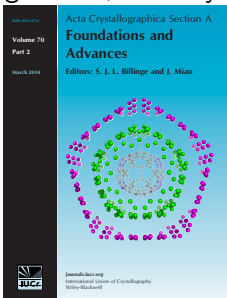
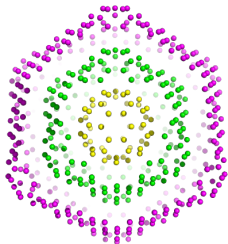
# New insight into RNA virus assembly

- There are **specific interactions** between **RNA** and coat protein (**CP**) given by **symmetry** axes
- Essential for **assembly** as only this RNA-CP interaction turns CP into **right geometric** shape for **capsid formation**
- The RNA forms a **Hamiltonian cycle** visiting each CP once – dictated by symmetry
- A **patent** for a new antiviral strategy (Reidun Twarock)



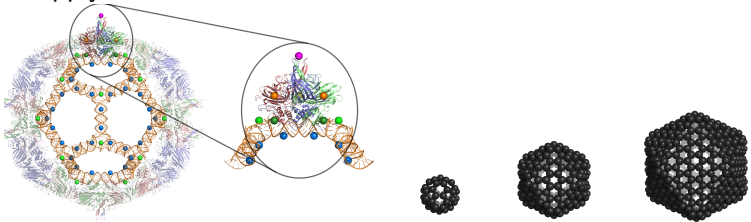
# Viruses and fullerenes – symmetry as a common thread?

- Get nested arrangements like Russian dolls: **carbon onions** (e.g. June: Nature 510, 250253)
- Potential to extend to **other known carbon onions** with different start configuration, chirality etc

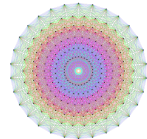
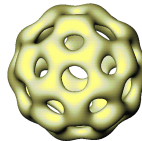
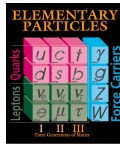
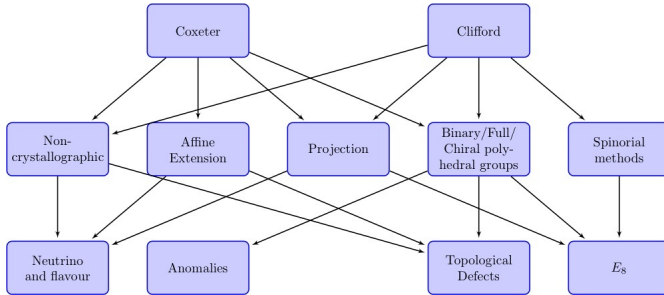


# Two major areas for Affine extensions of non-crystallographic Coxeter groups

- Non-compact symmetry that relates **different structural features** in the same polyhedral object
- **Novel symmetry principle** in Nature, shown that it seems to apply to at least **fullerenes** and **viruses**

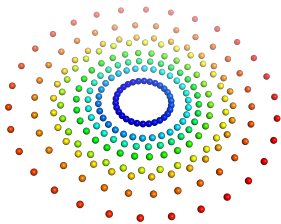
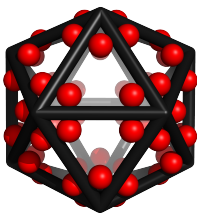


# Applications of these group structures in particle physics



# Purer Aspects of the Geometry

- Coxeter plane geometry, quaternionic representations, modular group etc
- Each 3D root system induces a 4D root system
- $H_3$  (icosahedral symmetry) induces the  $E_8$  root system
- Clifford algebra is a very natural framework for root systems and reflection groups



|         |         |
|---------|---------|
| $A_1^3$ | $A_1^4$ |
| $A_3$   | $D_4$   |
| $B_3$   | $F_4$   |
| $H_3$   | $H_4$   |

# Clifford Algebra and orthogonal transformations

- Form an algebra using the **Geometric Product** for two vectors

$$ab \equiv a \cdot b + a \wedge b$$

- **Inner product** is symmetric part  $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting  $a$  in  $b$  is given by  $a' = a - 2(a \cdot b)b = -bab$  ( $b$  and  $-b$  **doubly cover** the same reflection)
- Via **Cartan-Dieudonné** theorem any orthogonal (/conformal/modular) transformation can be written as **successive reflections**

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1 = \pm A x \tilde{A}$$

# Clifford Algebra of 3D

- E.g. **Pauli algebra** in 3D (likewise for **Dirac algebra** in 4D) is

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1 e_2 e_3\}}_{1 \text{ trivector}}$$

- We can **multiply together root vectors** in this algebra  $\alpha_i \alpha_j \dots$
- A general element has **8** components, **even** products (rotations/spinors) have **four** components:

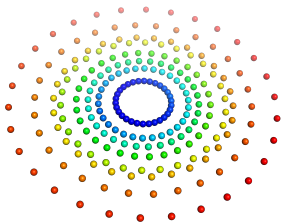
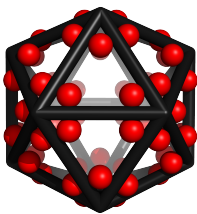
$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow R \tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

- So behaves as a **4D Euclidean** object – inner product

$$(R_1, R_2) = \frac{1}{2}(R_2 \tilde{R}_1 + R_1 \tilde{R}_2)$$

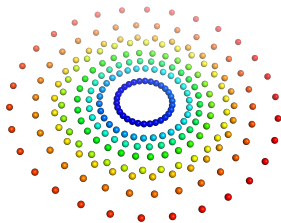
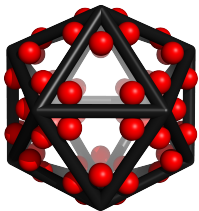
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| $H_3$   | $H_4$   |

Thank you!



|         |         |
|---------|---------|
| $A_1^3$ | $A_1^4$ |
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