A Clifford perspective on reflection groups, root systems and ADE correspondences

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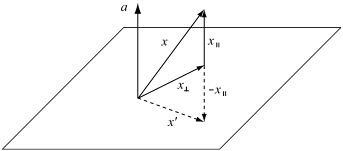
Reflections in Geometric Algebra



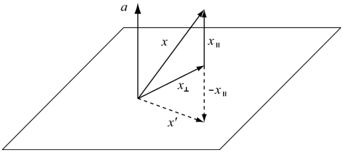
Reflections

$$x = x_{\perp} + x_{\parallel} \to x' = x_{\perp} - x_{\parallel} = x - 2x_{\parallel} = x - 2(x \cdot n)n$$

Vector space with an inner product

Why not work with the Clifford algebra?
Geometric product $ab \equiv a \cdot b + a \wedge b$
Inner product is the symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$

Reflections in Geometric Algebra



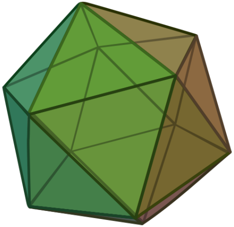
Reflections

$$x = x_\perp + x_\parallel \to x' = x_\perp - x_\parallel = x - 2x_\parallel = x - 2(x\cdot n)n = -nxn$$

Vector space with an inner product

Why not work with the Clifford algebra?
Geometric product $ab \equiv a\cdot b + a\wedge b$
Inner product is the symmetric part $a\cdot b = \frac{1}{2}(ab+ba)$

Groups of reflections (Coxeter groups)



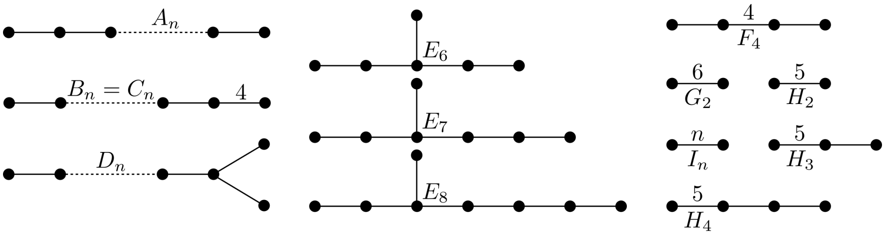
Reflection groups from generating reflections

$$\boxed{x' = -nxn} \rightarrow \boxed{x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1 =: \pm A x \tilde{A}}$$

Cartan-Dieudonné theorem

Any orthogonal transformation can be written as the product of successive reflections.

Classification of Euclidean reflection groups



Types

crystallographic (Weyl/Lie theory, A-G) vs non-crystallographic (I & H), simply-laced (ADE) etc

Groups and double covers

$$\boxed{x' = -nxn = -(-n)x(-n)}$$

So n and $-n$ doubly cover the same reflection.

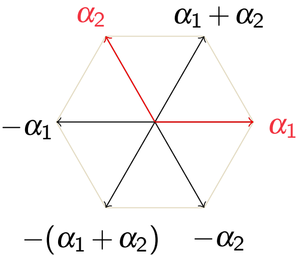
Reflection and rotation groups

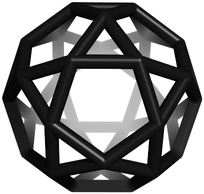
Reflection groups: subgroups of $O(n) \Rightarrow$ doubly covered in $\mathrm{Pin}(n)$
Rotation subgroups: subgroups of $SO(n) \Rightarrow$ doubly covered in $\mathrm{Spin}(n)$

Icosahedral groups

Reflection/Coxeter group H_3 in $O(3)$ of order 120.
Rotational subgroup A_5 in $SO(3)$ of order 60.
Pin group of H_3 in $\mathrm{Pin}(3)$ of order 240.
Binary icosahedral group $2I$ in $\mathrm{Spin}(3)$ of order 120 (or $SU(2)$ or quaternions).

Root systems





Root system Φ

A set of vectors α in a vector space with an inner product such that

1. $\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall\ \alpha \in \Phi}$
2. $\boxed{s_\alpha \Phi = \Phi \ \forall\ \alpha \in \Phi}$

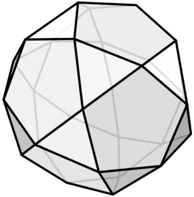
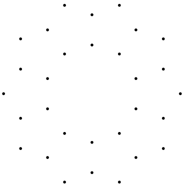
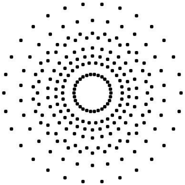
where the reflections and Coxeter element are $\boxed{s_\alpha : v \rightarrow s_\alpha(v) = -\alpha v \alpha}$ and $\boxed{w = s_1 \dots s_n}$

Vector space with an inner product

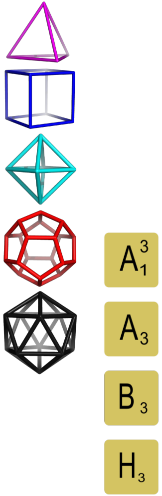
Why not work with the Clifford algebra?

The Coxeter Plane

- Every (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by projecting their root system onto the Coxeter plane

Platonic Solids



Platonic Solid	Group
Tetrahedron	A_3 A_1^3
Octahedron Cube	B_3
Icosahedron Dodecahedron	H_3

- Platonic Solids have been known for millennia
- Described by Coxeter groups

Clifford Algebra of 3D: the relation with 4D and 8D

- Clifford (Pauli) algebra in 3D is

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

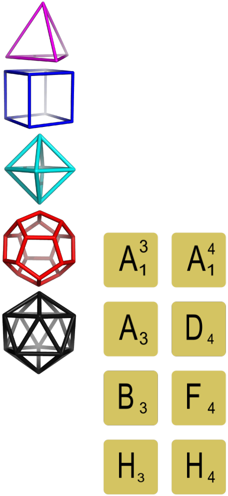
- We can multiply together root vectors in this algebra $\alpha_i\alpha_j\ldots$
- A general element has 8 components: 8D
- even products (rotations/spinors) have four components:

$$\boxed{R = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2 \Rightarrow R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

- So behaves as a 4D Euclidean object – inner product

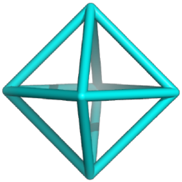
$$\boxed{(R_1, R_2) = \frac{1}{2}(R_2\tilde{R}_1 + R_1\tilde{R}_2)}$$

Platonic Solids



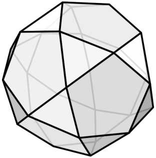
- Abundance of 4D root systems – exceptional
- Concatenating 3D reflections gives 4D Clifford spinors (binary polyhedral groups)
- These induce 4D root systems
  $R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow$
  $R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- This construction accidental to 3D perhaps explains the unusual abundance of 4D root systems

Spinors from reflections: easy example



- The 6 roots $(\pm 1, 0, 0)$ and permutations in $A_1 \times A_1 \times A_1$
- $\boxed{\pm e_1,\ \pm e_2,\ \pm e_3}$ generate group of 8 spinors
  $\boxed{\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1}$
- This is a discrete spinor group isomorphic to the quaternion group Q.
- As 4D vectors these are $(\pm 1, 0, 0, 0)$ and permutations, the 8 roots of $A_1 \times A_1 \times A_1 \times A_1$ (the 16-cell).

Spinors from reflections: icosahedral case



- The H_3 root system has 30 roots e.g. simple roots $\boxed{\alpha_1 = e_2, \alpha_2 = -\frac{1}{2}((\tau-1)e_1 + e_2 + \tau e_3) \text{ and } \alpha_3 = e_3}$.
- Subgroup of rotations A_5 of order 60 is doubly covered by 120 spinors of the form $\boxed{\alpha_1\alpha_2 = -\frac{1}{2}(1-(\tau-1)e_1e_2 + \tau e_2e_3)}$,
  $\boxed{\alpha_1\alpha_3 = e_2e_3}$ and $\boxed{\alpha_2\alpha_3 = -\frac{1}{2}(\tau-(\tau-1)e_3e_1 + e_2e_3)}$.
- These 120 spinors/roots constitute the H_4 root system

3D to 4D Induction (and 2D to 2D)

Induction theorem (3D to 4D)

The 3D root systems $(A_1 \times I_2(n), A_3, B_3, H_3)$ yield the 4D root systems $(I_2(n) \times I_2(n), D_4, F_4, H_4)$

Induction theorem (2D to 2D)

The 2D root systems $I_2(n)$ are self-dual i.e. only yield themselves $I_2(n)$

Arnold's Trinities

Arnold's observation that many areas of real mathematics can be complexified and quaternionified resulting in theories with a similar structure.

- The fundamental trinity is thus $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The projective spaces $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The spheres $(\mathbb{R}P^1 = S^1, \mathbb{C}P^1 = S^2, \mathbb{H}P^1 = S^4)$
- The Möbius/Hopf bundles $(S^1 \to S^1, S^4 \to S^2, S^7 \to S^4)$
- The Lie Algebras (E_6, E_7, E_8)
- The symmetries of the Platonic Solids (A_3, B_3, H_3)
- The 4D groups (D_4, F_4, H_4)
- New connections via this Clifford spinor construction

Platonic Trinities

- Arnold's connection between (A_3, B_3, H_3) and (D_4, F_4, H_4) is very convoluted and indirect
- Decomposition of the projective plane into Weyl chambers and Springer cones
- The number of Weyl chambers in each segment is $24 = 2(1+3+3+5), 48 = 2(1+5+7+11), 120 = 2(1+11+19+29)$
- Notice this miraculously is one less than the degrees of polynomial invariants $((2,4,4,6),(2,6,8,12),(2,12,20,30))$ of the Coxeter groups (D_4, F_4, H_4)
- The Clifford connection is much more direct
- But worth looking at these invariants as a clue

Trinity correspondence

- We now know the binary icosahedral group is essentially H_4, octahedral F_4 and tetrahedral D_4
- McKay correspondence relates even $SU(2)$ subgroups with ADE Lie algebras $(A_{2n-1}, D_{n+2}, E_6, E_7, E_8)$
- $(2n+2, 12, 18, 30)$ are the sum of the dimensions of the irreps and the ADE Coxeter number

2D/3D	$\lvert\Phi\rvert$	4D	G	$\sum d_i$	ADE	h
invariants						
A_3	12	D_4	$2T$	12		
B_3	18	F_4	$2O$	18		
H_3	30	H_4	$2I$	30		

More than a Trinity

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2D/3D	$\vert\Phi\vert$	4D	G	$\sum d_i$	ADE	h
Direct induction						
$A_1 \times I_2(n)$	$2n+2$	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$		
A_3	12	D_4	$2T$	12		
B_3	18	F_4	$2O$	18		
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But less than a McKay/ADE correspondence

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4D	G	$\sum d_i$	ADE	h
McKay/ADE			$\tilde{A}_{2n-1}$	$2n$
$I_2(n) \times I_2(n)$	Dic_n	$2n+2$	$\tilde{D}_{n+2}$	$2(n+1)$
D_4	$2T$	12	$\tilde{E}_6$	12
F_4	$2O$	18	$\tilde{E}_7$	18
H_4	$2I$	30	$\tilde{E}_8$	30

The countably infinite family $A_1 \times I_2(n)$ and Arnold's construction

For A_1^3

Can see immediately $8 = 2(1+1+1+1)$

For A_1^4

Simple roots $\alpha_1 = e_1$, $\alpha_2 = e_2$, $\alpha_3 = e_3$, $\alpha_4 = e_4$ give
$W = e_1 e_2 e_3 e_4 = (\cos\frac{\pi}{2} + \sin\frac{\pi}{2} e_1 e_2)(\cos\frac{\pi}{2} + \sin\frac{\pi}{2} e_3 e_4) = \exp(\frac{\pi}{2} e_1 e_2)\exp(\frac{\pi}{2} e_3 e_4)$
Gives exponents $(1,1,1,1)$ (from $h-1 = 2-1$)

The countably infinite family $A_1 \times I_2(n)$ and Arnold's construction

For $A_1 \times I_2(n)$

One gets the same decomposition $4n = 2(1+(n-1)+1+(n-1))$

For $I_2(n) \times I_2(n)$

Simple roots $\alpha_1 = e_1$, $\alpha_2 = -\cos\frac{\pi}{n}e_1 + \sin\frac{\pi}{n}e_2$, $\alpha_3 = e_3$, $\alpha_4 = -\cos\frac{\pi}{n}e_3 + \sin\frac{\pi}{n}e_4$ give $W = \exp\left(-\frac{\pi e_1 e_2}{n}\right)\exp\left(-\frac{\pi e_3 e_4}{n}\right)$
Gives exponents $(1,(n-1),1,(n-1))$

More than a Trinity

- We now know the binary icosahedral group is essentially H_4, octahedral F_4 and tetrahedral D_4
- McKay correspondence relates even $SU(2)$ subgroups with ADE Lie algebras $(A_{2n-1}, D_{n+2}, E_6, E_7, E_8)$
- $(2n+2, 12, 18, 30)$ are the sum of the dimensions of the irreps and the ADE Coxeter number

2D/3D	$\lvert\Phi\rvert$	4D	G	$\sum d_i$	ADE	h
Direct induction						
$A_1 \times I_2(n)$	$2n+2$	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$		
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The countably infinite family $I_2(n)$ and Arnold's construction

For $I_2(n)$

one gets the same decomposition $2n = 2(1 + (n-1))$

For $I_2(n)$

simple roots $\alpha_1 = e_1$, $\alpha_2 = -\cos\frac{\pi}{n}e_1 + \sin\frac{\pi}{n}e_2$ give
$W = \exp\left(-\frac{\pi e_1 e_2}{n}\right)$ gives exponents $(1, (n-1))$

The countably infinite families and Arnold's construction

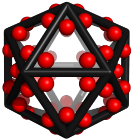
- So Arnold's initial hunch regarding the exponents extends in fact to my full correspondence
- McKay correspondence is a correspondence between even subgroups of $SU(2)$/quaternions and ADE affine Lie algebras
- In fact here get the even quaternion subgroups from 3D – link to ADE affine Lie algebras via McKay?

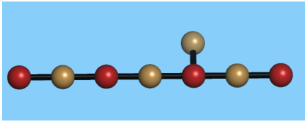
2D/3D, 2D/4D and ADE correspondences

- McKay correspondence relates even $SU(2)$ subgroups with ADE Lie algebras $(A_{2n-1}, D_{n+2}, E_6, E_7, E_8)$
- Induction theorem: get these as 2D/4D root systems $(I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$ from 2D/3D root systems $(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3)$
- $(2n, 2n+2, 12, 18, 30)$ are numbers of roots, the sum of the dimensions of the irreps and the ADE Coxeter number

2D/3D	$\|\Phi\|$	4D	G	$\sum d_i$	ADE	h
$I_2(n)$	$2n$	$I_2(n)$	C_{2n}	$2n$	$\tilde{A}_{2n-1}$	$2n$
$A_1 \times I_2(n)$	$2n+2$	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$	$\tilde{D}_{n+2}$	$2(n+1)$
A_3	12	D_4	$2T$	12	$\tilde{E}_6$	12
B_3	18	F_4	$2O$	18	$\tilde{E}_7$	18
H_3	30	H_4	$2I$	30	$\tilde{E}_8$	30

Is there a direct Platonic-ADE correspondence?





2D/3D		rot	ADE		legs
$I_2(n)$	n	n			
$A_1 \times I_2(n)$	n	$2, 2, n$			
A_3		$2, 3, 3$	E_6		$2, 3, 3$
B_3	4	$2, 3, 4$	E_7		$2, 3, 4$
H_3	5	$2, 3, 5$	E_8		$2, 3, 5$

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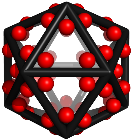
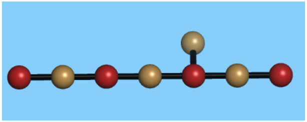
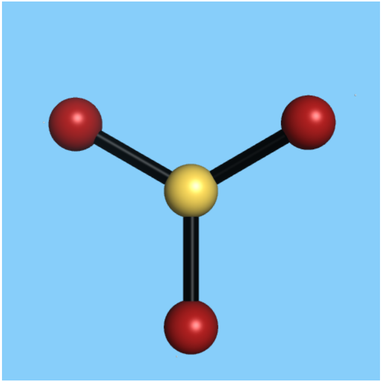




2D/3D		rot	ADE		legs
$I_2(n)$	n	n	A_n	$\cdots$	n
$A_1 \times I_2(n)$	n	$2, 2, n$	D_{n+2}	$\cdots$	$2, 2, n$
A_3		$2, 3, 3$	E_6		$2, 3, 3$
B_3	4	$2, 3, 4$	E_7		$2, 3, 4$
H_3	5	$2, 3, 5$	E_8		$2, 3, 5$

A Trinity of root system ADE correspondences

- 2D/3D root systems $(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3)$
- 2D/4D root systems $(I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$
- ADE root systems $(A_n, D_{n+2}, E_6, E_7, E_8)$

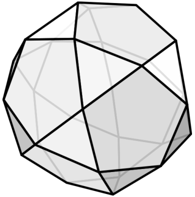


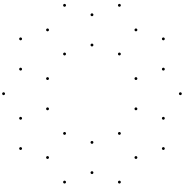
Coxeter Elements, Degrees and Exponents

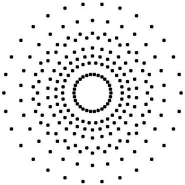
- Like the symmetric group, Coxeter groups can have invariant polynomials. Their degrees d are important invariants/group characteristics.
- E.g. each one has the Euclidean distance polynomial $x_1^2 + \cdots + x_n^2$ of degree 2
- Turns out that actually degrees d are intimately related to so-called exponents m $\boxed{m = d - 1}$, related to complex eigenvalues of the Coxeter element.

The Coxeter Plane

- Every (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by projecting their root system onto the Coxeter plane







Arnold's indirect connection between Trinities (A_3, B_3, H_3) and (D_4, F_4, H_4)

- Arnold had noticed a handwavey connection:
- Decomposition of 3D groups in terms of number of Springer cones matches what are essentially the exponents of the 4D groups:
- A_3: $24 = 2(1+3+3+5)$ – D_4: $(1,3,3,5)$
- B_3: $48 = 2(1+5+7+11)$ – F_4: $(1,5,7,11)$
- H_3: $120 = 2(1+11+19+29)$ – H_4: $(1,11,19,29)$

Coxeter Elements, Degrees and Exponents

Standard exposition

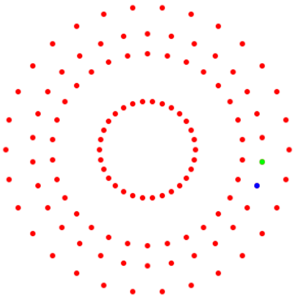
"In order to bring the eigenvalues of the Coxeter element w into the picture, we have to complexify the situation".

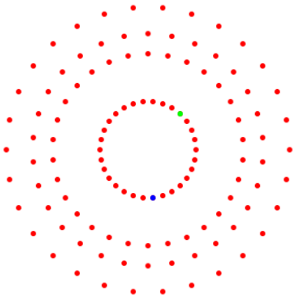
- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where m are called exponents
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again.
- In particular, 1 and $h-1$ are always exponents
- Turns out that actually exponents and degrees are intimately related ($\boxed{m=d-1}$). The construction is slightly roundabout but uniform, and uses the Coxeter plane.

4D case: H_4

- E.g. H_4 has exponents $1, 11, 19, 29$
- Coxeter versor decomposes into orthogonal components

$$\boxed{W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{11\pi}{30}IB_C\right)}$$





Arnold's indirect connection between Trinities

rank 4	exponents	W-factorisation
D_4	$1, 3, 3, 5$	$W = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{\pi}{2} IB_C\right)$
F_4	$1, 5, 7, 11$	$W = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} IB_C\right)$
H_4	$1, 11, 19, 29$	$W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} IB_C\right)$

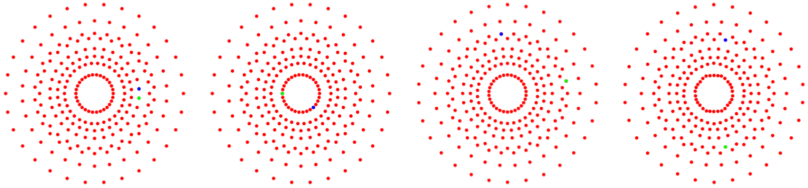
The remaining cases in the root system induction construction work the same way, not just this Trinity! So more general correspondence:

$$(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3) \to (I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$$

8D case: E_8

- E.g. H_4 has exponents $1, 11, 19, 29$, E_8 has $1, 7, 11, 13, 17, 19, 23, 29$
- Coxeter versor decomposes into orthogonal components

$$\boxed{W = \alpha_1 \dots \alpha_8 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{7\pi}{30} B_2\right) \exp\left(\frac{11\pi}{30} B_3\right) \exp\left(\frac{13\pi}{30} B_4\right)}$$



Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can factorise W into orthogonal (commuting/anticommuting) components
  $\boxed{W = \alpha_1 \dots \alpha_n = W_1 \dots W_n}$ with $\boxed{W_i = \exp(\pi m_i I_i / h)}$
- Here, I_i is a bivector describing a plane with $\boxed{I_i^2 = -1}$
- For v orthogonal to the plane descrbed by I_i we have
  $\boxed{v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i W_i v = v}$ so cancels out
- For v in the plane we have
  $\boxed{v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i / h) v}$
- Thus if we decompose W into orthogonal eigenspaces, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

Conclusions

- Clifford algebra provides a very general way of doing (reflection) group theory (Cartan-Dieudonné)
- Construction of the exceptional root systems from 3D root systems
- New and streamlined connections between algebraic concepts with increased understanding of the geometry
- More geometric approach to the geometry of the Coxeter plane, degrees and exponents
- Conceptual unification by working at the level of root systems (not group theory, Lie theory, polytopes etc)

Conclusions

Thank you!

Some papers with further details

- Dechant P-P, From the Trinity (A_3, B_3, H_3) to an ADE correspondence, Proceedings of the Royal Society A 474 (2220), 20180034
- Dechant P-P, The birth of E_8 out of the spinors of the icosahedron, PRSA 472(2185):20150504
- Dechant P-P, Clifford algebra is the natural framework for root systems and Coxeter groups. Group theory: Coxeter, conformal and modular groups. AACA 2017 1;27(1):17-31
- Dechant P-P, Clifford algebra unveils a surprising geometric significance of quaternionic root systems of Coxeter groups. AACA 2013 1;23(2):301-21.