A conformal geometric algebra construction of the modular group

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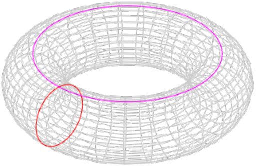
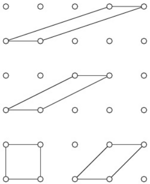
Overview

Torus



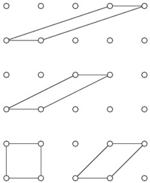
- A nice manifold, compact, 2 real-dimensional, or 1-complex dimensional
- The unique Calabi-Yau 1-fold
- The product of two circles: $T^2 = S^1 \times S^1$
- A torus is actually topologically flat, can cut open along the two circles to get a parallelogram

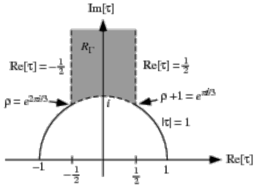
Torus embedding in the plane



- One can tile the plane with parallelograms in many ways (lattice)
- One can embed the torus if one picks such a lattice and periodically identifies opposite sides
- Considering the bottom left hand corner as the origin, and the top right hand corner as a number in the complex plane then this number is called the complex structure of the torus τ

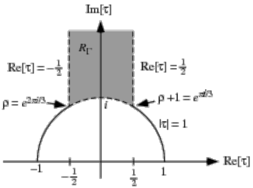
Symmetries of the embedding





- Many possible embeddings from the same torus: redundancy
- Winding around the torus more often $\boxed{\tau \to \tau + 1}$
- Flipping the long and the short side of the torus $\boxed{\tau \to -\frac{1}{\tau}}$

The modular group

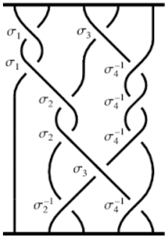


- The modular generators $T : \tau \to \tau + 1$, $S : \tau \to -\frac{1}{\tau}$ generate the modular group
- They satisfy the abstract relations $\boxed{\langle S, T | S^2 = I, (ST)^3 = I \rangle}$ ($T^\infty = I$)
- Keyhole fundamental region

The modular group as $SL(2,\mathbb{Z})$

- $SL(2,\mathbb{Z})$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\quad ad-bc=1$
- Subgroup of $SL(2,\mathbb{R})$, and the group of Moebius transformations of the plane, i.e. the 2D conformal group $PGL(2,\mathbb{C})$

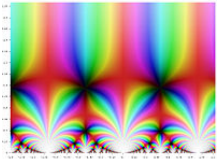
The braid group



- B_n: the group of braidings of n strands
- Presentation
  $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ (1 \leq i \leq n-2),$
  $\sigma_i \sigma_j = \sigma_j \sigma_i \ (|i-j| \geq 2) >$
- B_3 is a double cover of the modular group

Modular functions

$\mathbb{H} := \{\tau \in \mathbb{C} | Im(\tau) > 0\}$, $q = \exp(2i\pi\tau)$; $SL(2, \mathbb{R})$ action on $\mathbb{H}$:

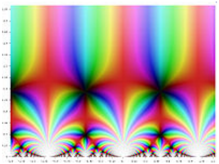


$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau+b}{c\tau+d}, \quad ad - bc = 1$$

Take G a discrete subgroup of $SL(2, \mathbb{R})$ commensurable with $SL(2, \mathbb{Z})$ then f_G is a modular function for G iff

- $f_G : \mathbb{H} \to \mathbb{C}$ is meromorphic
- $f_G(\tau) = f_G(\frac{a\tau+b}{c\tau+d}) \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$
- $f_G(A \cdot \tau) = \sum_{n=-\infty}^{\infty} b_n q^{n/N} \ \ \forall A \in SL(2, \mathbb{Z})$ and some N, b_n depending on A with $b_n = 0$ for $n < -M$, $M \in \mathbb{N}$

Modular forms

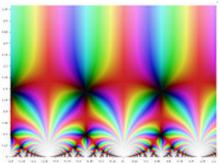


$\Phi(q)$ a modular form of $SL(2,\mathbb{Z})$ of weight k if

$$\Phi\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k\Phi(\tau)\ \ \forall\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z})$$

- Dedekind eta-function $\eta(q)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$
- Theta functions from an even unimodular lattice L: $\theta_L(z)=\sum_{\lambda\in L}\exp\pi i||\lambda||^2 z$
- Eisenstein series from a lattice Λ: $E_k(\Lambda)=\sum_{0\neq\lambda\in\Lambda}\lambda^{-k}$ a modular form of weight k

Weak Jacobi forms



weak Jacobi form of weight k and index m

$$\Phi(\frac{\mu}{c\mu+d},\frac{a\tau+b}{c\tau+d})=(c\tau+d)^{k}\exp\left(\frac{2i\pi mc\mu^2}{c\tau+d}\right)\Phi(\mu,\tau)$$

$$\forall\left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\in SL(2,\mathbb{Z}),\ \mu\in\mathbb{C},\tau\in\mathbb{H},z=\exp(2i\pi\mu),q=\exp(2i\pi\tau)$$

$$\Phi(\mu+a\tau+b,\tau)=\exp(-2i\pi m(a^2\tau+2b\mu))\Phi(\mu,\tau)\ \ \forall a,b\in\mathbb{Z}$$

Modular forms and number theory

- Number theorists are very active in modular things
- Connections with elliptic curves, Taniyama-Shimura, Andrew Wiles' proof of Fermat's theorem
- Extremely clever people, formidably hard (a lot worse than what's on the slides)

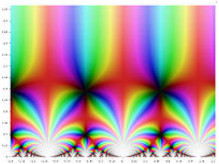
Moonshine

- Connection between two very different areas of Mathematics: finite simple groups and modular forms
- Moonshine = crazy, unlikely connection; insubstantial; illegal distilling of information from character table
- Monstrous = belonging to the Monster group, enormous, amazing

The Monster

- Finite simple groups: 18 series and 26 sporadic (exceptional)
- Monster: the largest sporadic group
- Order: $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \sim 8 \cdot 10^{53}$
- 194 irreducible representations: 1, 196883, 21296876, ...
- 20 sporadic groups are subquotients of the Monster: The Happy Family
- 6 are not: the Pariahs

The Klein j-invariant

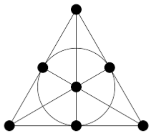


- j-invariant $j(\tau) = \frac{\Theta_{E_8}(\tau)^3}{\eta(\tau)^{24}} - 744$
- Hauptmodul for the genus 0 group $G = SL(2, \mathbb{Z})$
- Ogg: genus 0 iff p is $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71$ offering a bottle of Jack Daniel's whiskey
- Periodic: Fourier expansion coefficients wrt $q = e^{2\pi i \tau}$
- $\boxed{j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots}$

Monstrous Moonshine

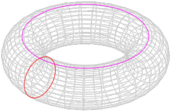
- Mysterious connection between two very different areas (discrete and non-discrete) of Mathematics noticed by John McKay 1978
- Monster $\boxed{1,\ 196883,\ 21296876,\ldots}$
- Klein $\boxed{j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots}$
- $$\boxed{196884 = 196883 + 1,\ 21493760 = 21296876 + 196883 + 1,\ \ldots}$$
- Conway, Norton conjectured, Atkin, Fong, and Smith showed a moonshine module exists in 1980 (vertex operator algebra)
- Constructed by Richard Borcherds in 1992

The Mathieu group M_{24}



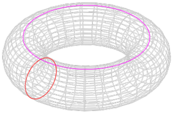
- One of 5 sporadic groups (first to be discovered) named after Mathieu: $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$
- Multiply transitive permutation groups
- $244823040 = 3 \cdot 16 \cdot 20 \cdot 21 \cdot 22 \cdot 23 \cdot 24 = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
- Arises as automorphism group of the extended binary Golay code, Leech lattice or Steiner system $S(5, 8, 24)$
- 26 irreps $1, 23, 45, 231, 252, 253, 483, 770, 990, 1035, 1265, 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395$

K3 – Kähler, Kummer, Kodaira (mountain K2 in Kashmir, André Weil)



- 2 complex-dimensional manifold
- $T^2 \times T^2$ and K3 are the only Calabi-Yau 2-folds
- Kähler manifold, Ricci-flat, compact
- Often used in string theory compactifications along with T^2
- From orbifold resolutions, Kummer surfaces etc – all diffeomorphic

String Theory



- A supersymmetric point particle evolving along a closed loop in space-time
- A closed superstring sweeps out a 2D torus
- Index of Dirac operator and its stringy generalisation Ramond operator
- Compactifications on tori and K3 manifolds
- Interesting (roughly) modular quantities and properties arise in string theory: e.g. $N = 4$ characters, elliptic genus = counts different states

K3 elliptic genus

- Known since 80s, innocuous enough looking

$$E_{K3}(z,q) = 8\left(\frac{\theta_2^2(z,q)}{\theta_2^2(1,q)} + \frac{\theta_3^2(z,q)}{\theta_3^2(1,q)} + \frac{\theta_4^2(z,q)}{\theta_4^2(1,q)}\right)$$

- A weak Jacobi form of weight $k = 0$ and index $m = 1$
- Special values give Euler characteristic, $\hat{A}$ Dirac index, σ Hirzebruch signature
- If rewrite this in terms of $N = 4$ characters of a specific string theoretic non-linear sigma model describing superstring propagation on a K3 surface (!) one gets

$$\boxed{E_{K3}(\tau,z) = -2Ch(0;\tau,z) + 20Ch(1/2;\tau,z) + e(q)Ch(\tau,z)}$$

- Coefficients in the q-series are

$$\boxed{e(q) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + \dots}$$

Mathieu Moonshine – Eguchi, Ooguri, Tachikawa 2010

- Similar Moonshine phenomenon connecting finite simple groups and modular forms (ish..)
- elliptic genus of an $\mathcal{N}=4$ SCFT compactified on a K3-surface (Taormina, Eguchi 80s)
- Finite simple group: Mathieu M_{24} $45, 231, 770, 2277, 5796\dots$
- Elliptic genus is
$$\boxed{E_{K3}(\tau,z) = -2Ch(0;\tau,z) + 20Ch(1/2;\tau,z) + e(q)Ch(\tau,z)}$$
- All the coefficients in the q-series $\boxed{e(q) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + \dots}$ are twice the dimension of some M_{24} irrep

Mathieu Moonshine – state of the field

- Generally pretty lost what's going on
- Partial results, e.g. proving coefficients are positive even integers, dimensions of M_{24} irreps etc
- Nothing in the string theory actually can have full M_{24} symmetry!?
- Elliptic genus is invariant under surfing across different Kummer surfaces despite counting different states in different theories etc e.g. given by symmetry group of D_4 lattice i.e. binary tetrahedral group
- Generalisation umbral moonshine connecting M_{24} to 23 other symmetry groups of Niemeier lattices

Mathieu Moonshine – state of the field

- Number theorists chime in: weakly holomorphic mock modular form of weight $1/2$ on $SL(2,\mathbb{Z})$ with shadow $\eta(q)^3$. All extremely technical
- My construction of exceptional root systems from Thursday ties in with these things: lattices, symmetry groups of Kummer surfaces, McKay correspondence, Trinities, Moonshine
- New approach to modular symmetry?

Overview

Clifford Algebra and orthogonal transformations

- Inner product is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting a in b is given by $\boxed{a' = a - 2(a \cdot b)b = -bab}$ (b and $-b$ doubly cover the same reflection)
- Via Cartan-Dieudonné theorem any orthogonal (/conformal/modular) transformation can be written as successive reflections

$$\boxed{x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1} = \pm A x \tilde{A}$$

- The conformal group $C(p, q) \sim SO(p+1, q+1)$ so can use these for translations, inversions etc as well

Conformal Geometric Algebra

- Go to $e_0, e_1, e_2, e_3, e, \bar{e}$, with $e_0^2 = 1, e_i^2 = -1, e^2 = 1, \bar{e}^2 = -1$
- Define two null vectors $n \equiv e + \bar{e},\ \bar{n} \equiv e - \bar{e}$
- Can embed the 4D vector $x = x^\mu e_\mu = te_0 + xe_1 + ye_2 + ze_3$ as a null vector in 6D (also normalise $\hat{X} \cdot e = -1$)

$$\hat{X} = \frac{1}{\lambda^2 - x^2}(x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

- So neat thing is that conformal transformations are now done by rotors (except inversion which is a reflection) – distances are given by inner products

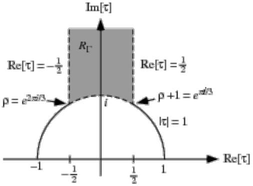
Conformal Transformations in CGA

$$\hat{X} = \frac{1}{\lambda^2 - x^2}(x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

- Reflection: spacetime $F(-axa) = -aF(x)a$
- Rotation: spacetime $F(Rx\tilde{R}) = RF(x)\tilde{R}$, $R = \exp(\frac{ab}{2\lambda})$
- Translation: $F(x+a) = R_T F(x)\tilde{R}_T$ for $R_T = \exp(\frac{na}{2\lambda}) = 1 + \frac{na}{2\lambda}$
- Dilation: $F(e^{\alpha}x) = R_D F(x)\tilde{R}_D$ for $R_D = \exp(\frac{\alpha}{2\lambda}e\bar{e})$
- Inversion: Reflection in extra dimension e: $F(\frac{x}{x^2}) = -eF(x)e$ sends $n \leftrightarrow \bar{n}$
- Special conformal transformation: $F(\frac{x}{1+ax}) = R_S F(x)\tilde{R}_S$ for $R_S = R_I R_T R_I$

Overview

Modular group



- Modular generators: $T: \tau \to \tau + 1$, $S: \tau \to -1/\tau$
- $\boxed{\langle S, T | S^2 = I, (ST)^3 = I \rangle}$ CGA: $\boxed{R_Y X \tilde{R}_Y}$
- CGA: $\boxed{T_X = \exp\left(\frac{n e_1}{2}\right) = 1 + \frac{n e_1}{2}}$ and $\boxed{S_X = e_1 e}$ (slight issue of complex structure τ = complex number, not vector in the 2D real plane so map $e_1 : x_1 e_1 + x_2 e_2 \leftrightarrow x_1 + x_2 e_1 e_2 = x_1 + i x_2$)
- $\boxed{(S_X T_X)^3 = -1}$ and $\boxed{S_X^2 = -1}$
- So a 3-fold and a 2-fold rotation in conformal space

Braid group

- $(S_X T_X)^3 = -1$ and $S_X^2 = -1$ is inherently spinorial
- Of course Clifford construction gives a double cover
- The braid group is a double cover
- So Clifford construction gives the braid group double cover of the modular group
- $\sigma_1 = \tilde{T}_X$ and $\sigma_2 = T_X S_X T_X$ satisfying $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ $(= S_X)$

Braid group

- $(S_X T_X)^3 = -1$ and $S_X^2 = -1$ is inherently spinorial
- Of course Clifford construction gives a double cover
- The braid group is a double cover
- So Clifford construction gives the braid group double cover of the modular group
- $\sigma_1 = \tilde{T}_X = \exp(-ne_1/2)$ and $\sigma_2 = T_X S_X T_X = \exp(-\bar{n}e_1/2)$ satisfying $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ $(= S_X)$
- Might not be known?

Where to go from here?

- Routinely do Clifford complex analysis in the plane
- Could look at meromorphic functions
- Look at functions in the complex plane in CGA representation
- Consider modular symmetry in this setup: modular functions, modular forms, weak Jacobi forms etc
- Perhaps the spinorial approach opens up new techniques to deal with the formidable algebra?