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Root systems & Clifford algebras: from symmetries of viruses to E_8 & ADE correspondences

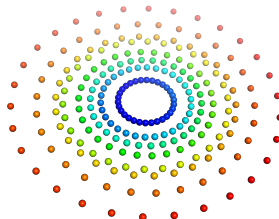
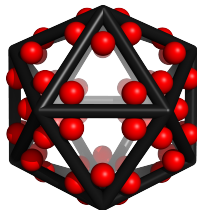
Pierre-Philippe Dechant

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Nankai Symposium on Physics, Geometry, and Number Theory
August 4, 2017

Main results

- New **affine symmetry principle** for viruses and fullerenes
- H_3 (**icosahedral symmetry**) induces the E_8 root system
- Each **3D** root system **induces a 4D** root system
- This **correspondence** extends to various **ADE** correspondences



Reflection groups: a new approach

- Work at the level of **root systems** (which define reflection groups)
- Interested in **non-crystallographic** root systems e.g. viruses, fullerenes etc. But: no Lie algebra, so conventionally less studied
- **Clifford algebra** is a uniquely suitable framework for reflection groups/root systems: **reflection formula**, spinor **double covers**, **complex/quaternionic quantities** arising as **geometric objects**

1 Viruses, root systems and affine extensions (with R. Twarock)

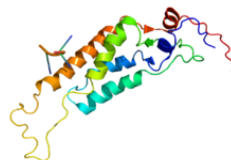
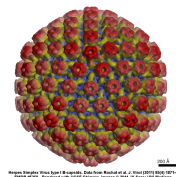
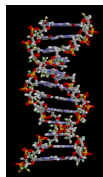
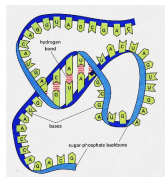
- Viruses
- Root systems
- Affine extensions
- Fullerenes

2 Clifford algebras, exceptional root systems and ADE correspondences

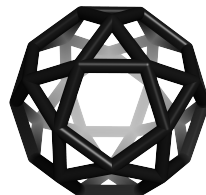
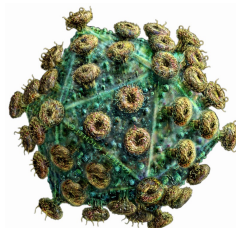
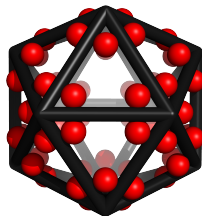
- Clifford basics
- E_8 from the icosahedron
- 3D to 4D spinor induction
- McKay/ADE correspondences

What is a Virus?

- Transported piece of **genetic information** that e.g. can run a programme in a host cell
- **Genome**: RNA or DNA
- Fragile – needs to be protected by a **protein** shell: **capsid**
- **Gene** \rightarrow mRNA \rightarrow **protein** (transcription and translation)
- Each **protein** = amino acid chain folds into a 3D shape: one **geometric building block**



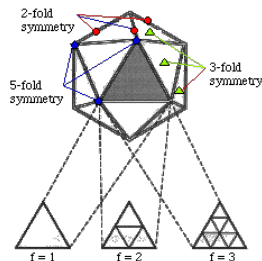
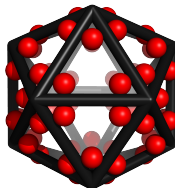
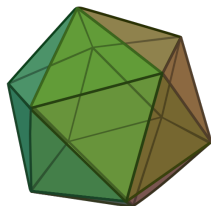
Watson and Crick: The Icosahedron



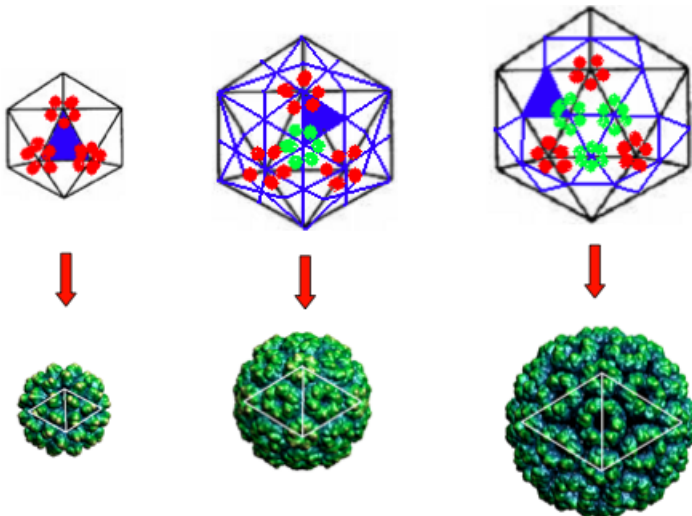
- **Crick&Watson**: Genetic economy \rightarrow symmetry \rightarrow icosahedral is largest
- **Rotational** icosahedral group is $I = A_5$ of order 60
- **Full** icosahedral group is the **Coxeter group** H_3 of order 120 (including reflections/inversion); generated by the **root system icosidodecahedron**

Caspar and Klug: Triangulations

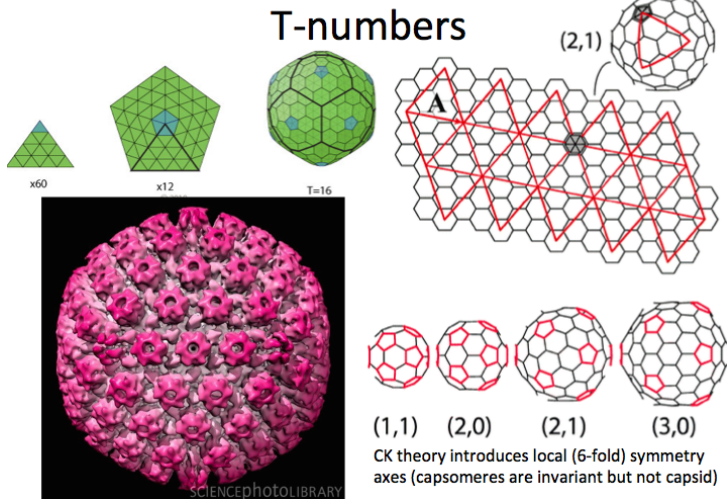
- Mathematical upper limit of **60** for **equivalent** subunits, but biologically want to do better!
- Gene \rightarrow can already make a **triangle** \rightarrow might as well make **many**!
- Caspar-Klug ideas of quasi-equivalence and **triangulations**



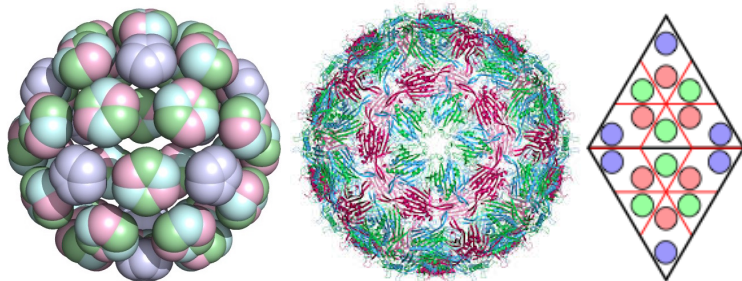
Viruses: Caspar-Klug triangulations



Viruses: Caspar-Klug triangulations

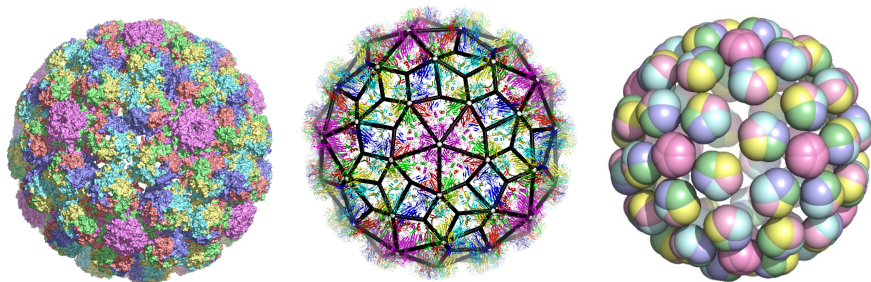


Icosahedral viruses: triangulations and other quasi-equivalent tilings



Two **viral surface** layouts: a $T = 4$ **triangulation** (e.g. HBV) and a **rhombus** tiling (MS2) for a pseudo $T = 3$ **triangulation**

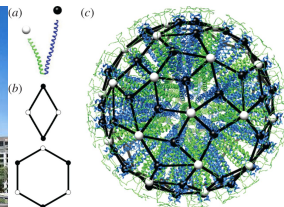
Icosahedral viruses: non-quasiequivalent tilings – Penrose



More general icosahedral tilings: Cryo-EM **reconstruction** of HPV, a **kite-rhombus** tiling and a pseudo $T = 7$ **triangulation**.

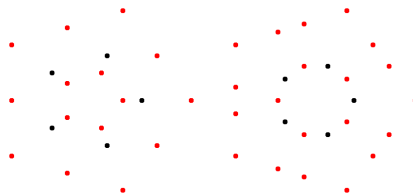
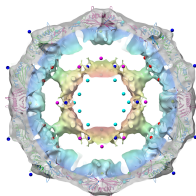
Other applications

- **Architecture**: Buckminster Fuller geodesic domes
- The architectural analogue of the kite-rhombus tiling: the new **Amazon HQ**
- **Nanoparticles** based on kite-rhombus tiling and local interaction rules

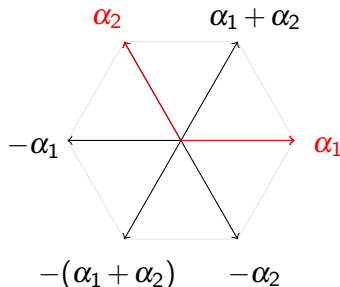


Motivation: Viruses

- Improves the limit to $60T$, but only in terms of **surface structures** (12 pentagons and rest hexagons).
- Making the symmetry non-compact might allow more general symmetry, **simultaneously constraining** different 'radial levels'
- Non-compact generator is a **translation** – motivates looking into **affine extensions** of icosahedral symmetry
- There is an **inherent length scale** in the problem – given by size of nucleic acid/protein molecules



Root systems



reflection/Coxeter groups

Root system Φ : set of vectors α in a **vector space** with an **inner product** such that

1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$
2. $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

Simple roots: express every element of Φ via a **\mathbb{Z} -linear combination**.

$$s_\alpha : v \rightarrow s_\alpha(v) = v - 2 \frac{(v|\alpha)}{(\alpha|\alpha)} \alpha$$

Cartan Matrices

Cartan matrix of α_i s is $A_{ij} = 2 \frac{(\alpha_i | \alpha_j)}{(\alpha_i | \alpha_i)} = 2 \frac{|\alpha_j|}{|\alpha_i|} \cos \theta_{ij}$

$$A_2: A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal, simple link = roots at $\frac{\pi}{3}$, link with label m = angle $\frac{\pi}{m}$.

$$A_3 \circ - \circ - \circ$$

$$B_3 \circ - \overset{4}{\circ} - \circ$$

$$H_3 \circ - \overset{5}{\circ} - \circ$$

$$I_2(n) \circ - \overset{n}{\circ}$$

Lie groups to Lie algebras to Coxeter groups to root systems

- **Lie group**: manifold of continuous symmetries (gauge theories, spacetime)
- **Lie algebra**: infinitesimal version near the identity
- Non-trivial part is given by a **root lattice**
- **Weyl** group is a **crystallographic** Coxeter group:
 $A_n, B_n/C_n, D_n, G_2, F_4, E_6, E_7, E_8$ generated by a **root system**.
- So via this route root systems are **always** crystallographic.
Neglect non-crystallographic root systems $I_2(n), H_3, H_4$.

Affine extensions

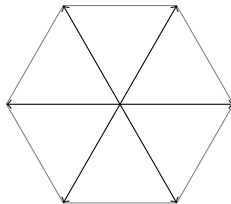
An **affine Coxeter group** is the extension of a Coxeter group by an **affine reflection in a hyperplane not containing the origin** $s_{\alpha_0}^{aff}$ whose geometric action is given by

$$s_{\alpha_0}^{aff} v = \alpha_0 + v - \frac{2(\alpha_0 | v)}{(\alpha_0 | \alpha_0)} \alpha_0$$

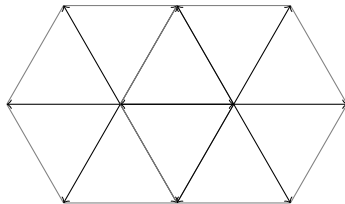
Non-distance preserving: includes the **translation generator**

$$Tv = v + \alpha_0 = s_{\alpha_0}^{aff} s_{\alpha_0} v$$

Affine extensions – A_2

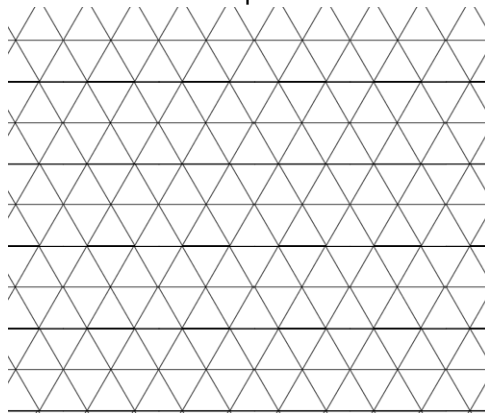


Affine extensions – A_2

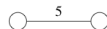
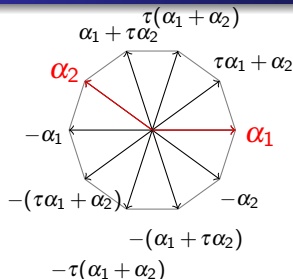


Affine extensions – A_2

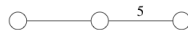
Affine extensions of crystallographic Coxeter groups lead to a **tessellation** of the plane and a **lattice**.



Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$



$$A = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate **rotational symmetries of order 5**
linear combinations now in the **extended integer ring**

$$\mathbb{Z}[\tau] = \{a + \tau b \mid a, b \in \mathbb{Z}\} \quad \text{golden ratio}$$

$$\tau = \frac{1}{2}(1 + \sqrt{5}) = 2 \cos \frac{\pi}{5}$$

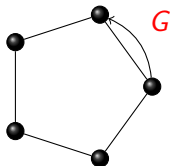
$$x^2 = x + 1$$

$$\tau' = \sigma = \frac{1}{2}(1 - \sqrt{5}) = 2 \cos \frac{2\pi}{5}$$

$$\tau + \sigma = 1, \tau\sigma = -1$$

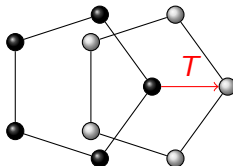
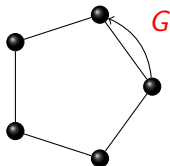
Affine extensions of non-crystallographic root systems?

Unit translation along a vertex of a unit pentagon



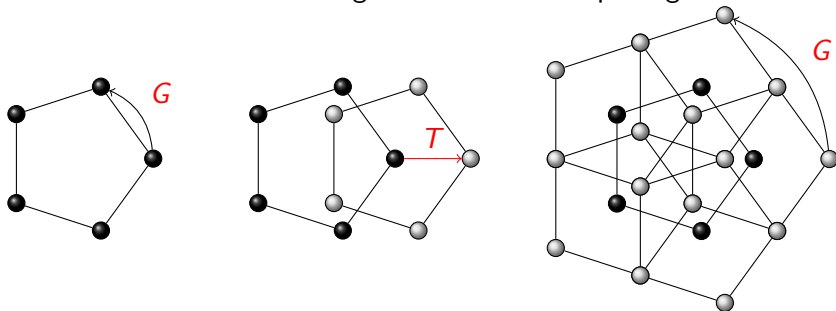
Affine extensions of non-crystallographic root systems?

Unit translation along a vertex of a unit pentagon



Affine extensions of non-crystallographic root systems?

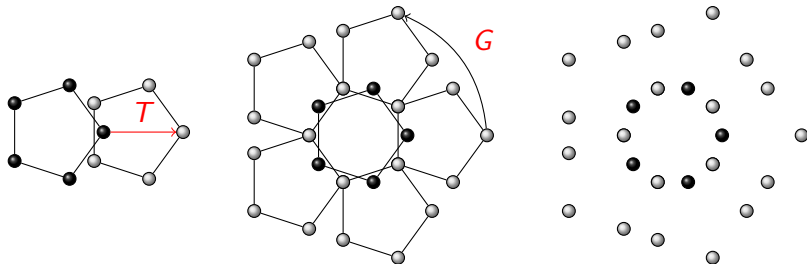
Unit translation along a vertex of a unit pentagon



A **random** translation would give 5 secondary pentagons, i.e. 25 points. Here we have **degeneracies** due to 'coinciding points'.

Affine extensions of non-crystallographic root systems?

Translation of length $\tau = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ (golden ratio)

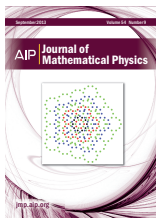


Cartoon version of a **virus** or **carbon onion**. Would there be an **evolutionary benefit** to have more than just compact symmetry?

The problem has an **intrinsic length scale**.

Affine extensions of non-crystallographic Coxeter groups

- 2D and 3D **point arrays** for applications to viruses, fullerenes, quasicrystals, proteins etc
- Two complementary ways** to construct these



Know your onions

Acta Cryst. A 70, 162-167 (2014)

Many viruses have icosahedral symmetry. So do certain 'carbon onions' — Russian doll-like arrangements of nested fullerenes. Pierre-Philippe Dechant and colleagues argue that viruses and carbon onions share the same formation principle: affine symmetry. Imagine a set of points lying on the vertices of a regular pentagon. Duplicate the set, and translate it, then repeatedly rotate the combined set over 72° about the midpoint of the original pentagon. This results in a new set of points obeying five-fold symmetry, yet with a 2D shell structure that is more complex than that of the pentagon. A similar 'affinization' of the 3D icosahedral group results in a set of points that are nodes in the highly complex protein network structure of, for example, the Penicillin virus.

Dechant *et al.* found that affine symmetry explains the structure of experimentally observed carbon onions — a non-trivial result given that all carbon atoms in each of the nested fullerene molecules must be three-connected, that is, bound to three neighbouring carbons. In particular, they identified the extended group that, starting from buckminsterfullerene (the 'buckyball'), generates the onion $C_{60}@C_{60}@C_{60}$.

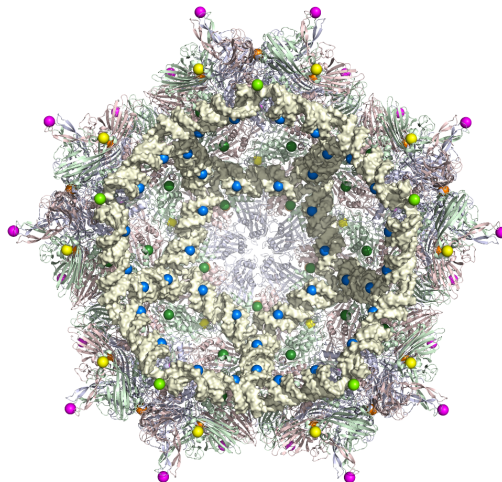
well known effect for photons, and it turns out to hold for other quantum particles too. James Fokas and colleagues have performed the Hong–Ou–Mandel quantum interference experiment using plasmons, which are quantized surface plasma waves. Pairs of photons are fed into a specially designed plasmonic waveguide that mixes the paths of the light-excited surface plasmons in the same way as a beam splitter. The outcome is converted back into photons and measured by two detectors. As in the purely photonic case, the characteristic dip in coincidence rate is shown, showing that the photons remain indistinguishable when they are converted into plasmons and interfere.

Written by May Chiao, Miki Georgiou, Abigail Kopper, Bart Verbrink and Adam Wright

NATURE PHYSICS | VOL 10 | APRIL 2014 | www.nature.com/naturephysics

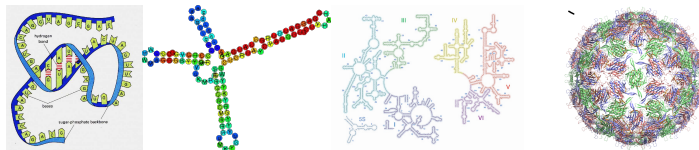
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Use in Mathematical Virology

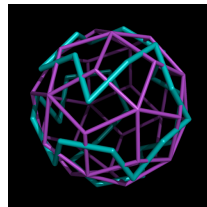
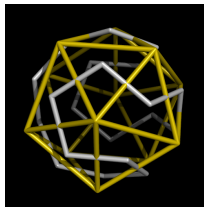
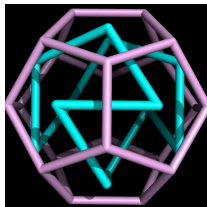


New insight into RNA virus assembly

- There are **specific interactions** between **RNA** and coat protein (**CP**) given by icosahedral **symmetry** axes
- Essential for **assembly**, as only this RNA-CP interaction turns CP into **right geometric shape** for **capsid formation**
- The RNA forms a **Hamiltonian cycle** visiting each CP once – dictated by symmetry



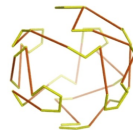
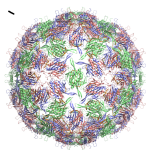
Hamiltonian cycles on icosahedral solids



- So **interaction contacts** are given by the **symmetry**
- Orbits of the interaction points have to be visited by the **RNA exactly once**
- Even the RNA has an **icosahedrally ordered component**
- **Hamiltonian cycles** for dodecahedron, icosahedron and rhombic triacontahedron

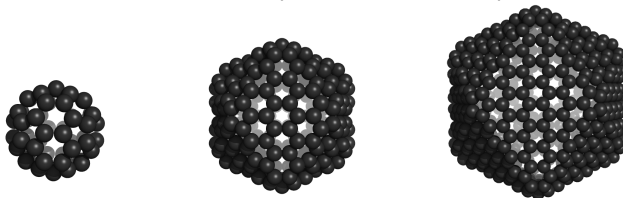
New insight into RNA virus assembly

- More realistic examples for MS has **60 vertices** with 41,000 paths
- The RNA is actually **circularised** by Maturation Protein: only **66 cycles**
- With thermodynamical **assembly kinetics** and **5-fold averaging** experiments **uniquely** identified an **evolutionarily conserved** cycle
- **Patents** for new **antiviral strategies** and **virus-like nanoparticles** e.g. for **drug delivery** (Twarock group)



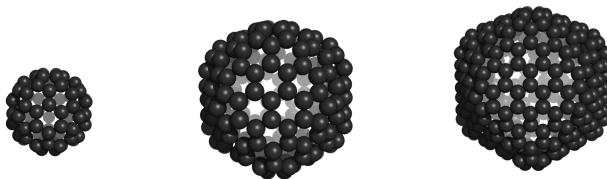
Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach: **carbon onions** ($C_{60} - C_{240} - C_{540}$)



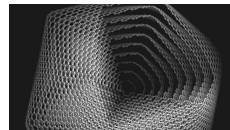
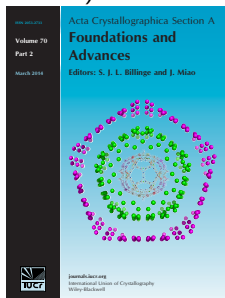
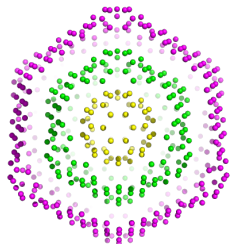
Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach: **carbon onions** ($C_{80} - C_{180} - C_{320}$)



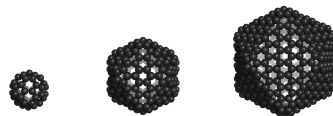
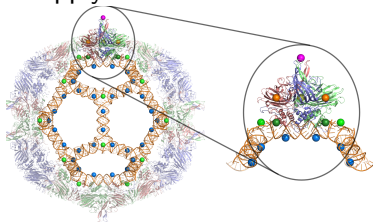
Viruses and fullerenes – symmetry as a common thread?

- Get nested arrangements like Russian dolls: **carbon onions** (e.g. Nature 510, 250253)



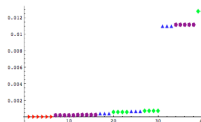
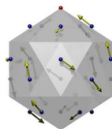
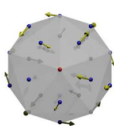
Two major areas for affine extensions of non-crystallographic Coxeter groups

- Non-compact symmetry that relates **different structural features** in the same polyhedral object when there is an additional **length scale**
- **Novel symmetry principle** in Nature, shown that it seems to apply to at least **fullerenes** and **viruses**



Vibrations of capsids and fullerenes

- Normal modes/vibrations of icosahedral capsids given by representation theory of the icosahedral group
- E.g. $\Gamma_{\text{Icos}}^{\text{disp}} = \Gamma^1 + 3\Gamma^3 + \Gamma^{3'} + 2\Gamma^4 + 3\Gamma^5$
- Pioneered by Anne Taormina, Kasper Peeters and Francois Englert



J	1	$20C_2$	$12C_2$	$12C_5$	$12C_5'$
1	1	1	1	1	1
3	3	0	-1	τ	τ
3	3	0	-1	τ	τ
4	4	1	0	-1	-1
5	5	-1	1	0	0
Icos perm	12	0	0	2	2
$\chi_{\text{Icos}}^{\text{new}}$	36	0	0	2τ	2τ
Dodec	20	2	0	0	0
perm					
$\chi_{\text{Dodec}}^{\text{new}}$	60	0	0	0	0
IHD perm	30	0	2	0	0
$\chi_{\text{IHD}}^{\text{new}}$	90	0	-2	0	0
tr	300	0	0	0	0
perm					
$\chi_{\text{tr}}^{\text{new}}$	180	0	0	0	0

- 1 Viruses, root systems and affine extensions (with R. Twarock)
 - Viruses
 - Root systems
 - Affine extensions
 - Fullerenes
- 2 Clifford algebras, exceptional root systems and ADE correspondences
 - Clifford basics
 - E_8 from the icosahedron
 - 3D to 4D spinor induction
 - McKay/ADE correspondences

Clifford Algebra and orthogonal transformations

- **Reflection** group setting: **vector space** with an **inner product**
- Form an algebra using the product between two vectors

$$ab \equiv a \cdot b + a \wedge b$$

- **Inner product** is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting x in n is given by $x' = x - 2(x \cdot n)n = -nxn$ (n and $-n$ **doubly cover** the same reflection)
- Via **Cartan-Dieudonné** theorem any orthogonal (/conformal/modular) transformation can be written as **successive reflections**

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1 = \pm A x \tilde{A}$$

Clifford Algebra of 3D: geometric objects and algebraic relations

- The Clifford algebra in 3D is

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1 e_2 e_3\}}_{1 \text{ trivector}}$$

- Parallel vectors **commute**, orthogonal vectors **anticommute**
- Note $e_1 e_2$ etc square to -1 i.e. are imaginaries representing xy-plane etc i.e. are **complex**
- $\{1, e_1 e_2, e_2 e_3, e_3 e_1\}$ together satisfy **quaternionic** relations

Clifford Algebra of 3D: the relation with 4D and 8D

- E.g. **Pauli algebra** in 3D (likewise for **Dirac algebra** in 4D) is

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1 e_2 e_3\}}_{1 \text{ trivector}}$$

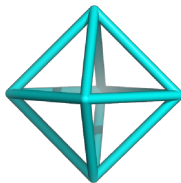
- We can **multiply together root vectors** in this algebra $\alpha_i \alpha_j \dots$
- A general element has **8** components (8D vector space), **even** products (rotations/spinors) have **four** components (4D subspace):

$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow R \tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

- So behaves as a **4D Euclidean** object – inner product

$$(R_1, R_2) = \frac{1}{2}(R_2 \tilde{R}_1 + R_1 \tilde{R}_2)$$

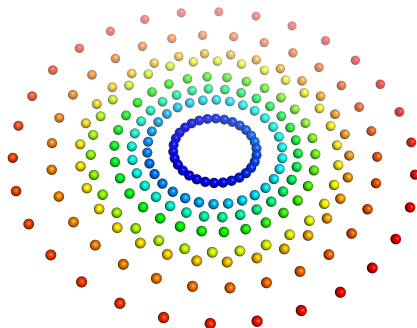
Spinors from reflections



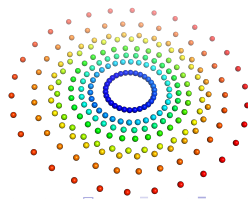
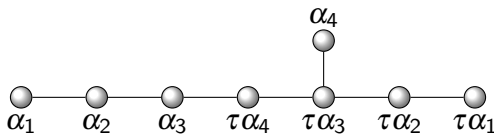
- The 6 **roots** $(\pm 1, 0, 0)$ and permutations of $A_1 \times A_1 \times A_1$ generate 8 **spinors**:
- $\boxed{\pm e_1, \pm e_2, \pm e_3}$ give the 8 spinors $\boxed{\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1}$
- This is a **discrete spinor group** isomorphic to the **quaternion** group Q .
- As **4D vectors** these are $(\pm 1, 0, 0, 0)$ and permutations, the 8 **roots** of $A_1 \times A_1 \times A_1 \times A_1$ (the 16-cell).

Exceptional E_8 (projected into the Coxeter plane)

E_8 root system has 240 roots, H_3 has order 120



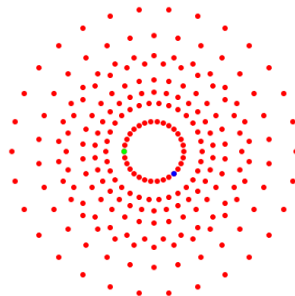
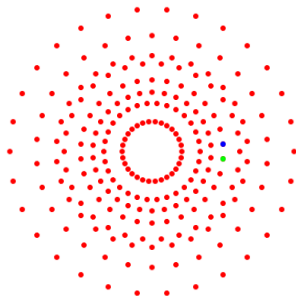
- Order 120 group H_3 doubly covered by 240 (s)pinors in 8D space
- With (somewhat counterintuitive) reduced inner product this gives the E_8 root system
- E_8 is actually hidden within 3D geometry!



Real Clifford geometry: E_8

- E_8 has **exponents** 1, 7, 11, 13, 17, 19, 23, 29, seen as **complex** eigenvalues of the **Coxeter element**
- In Clifford algebra, Coxeter element factorises

$$W = \alpha_1 \dots \alpha_8 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{7\pi}{30} B_2\right) \exp\left(\frac{11\pi}{30} B_3\right) \exp\left(\frac{13\pi}{30} B_4\right)$$



Induction Theorem – root systems

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- In 2D, the space of spinors is also 2D and the root systems are self-dual under an analogous construction

Trinity of 4D Exceptional Root Systems

- **Exceptional** phenomena: D_4 (**triality**, important in string theory), F_4 (**largest lattice symmetry** in 4D), H_4 (**largest non-crystallographic symmetry**); **Exceptional** D_4 and F_4 arise from **series** A_3 and B_3

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$	
A_3		$2T$	D_4	
B_3		$2O$	F_4	
H_3		$2I$	H_4	

Arnold's indirect connection between Trinities (A_3, B_3, H_3) and (D_4, F_4, H_4)

- Arnold had noticed a handwavey connection:
- Decomposition of 3D groups in terms of number of **Springer cones** matches what are essentially the **exponents** of the 4D groups:
- A_3 : $24 = 2(1 + 3 + 3 + 5) - D_4$: $(1, 3, 3, 5)$
- B_3 : $48 = 2(1 + 5 + 7 + 11) - F_4$: $(1, 5, 7, 11)$
- H_3 : $120 = 2(1 + 11 + 19 + 29) - H_4$: $(1, 11, 19, 29)$

Arnold's indirect connection between Trinities

rank 4	exponents	W-factorisation
D_4	1, 3, 3, 5	$W = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{\pi}{2} I B_C\right)$
F_4	1, 5, 7, 11	$W = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} I B_C\right)$
H_4	1, 11, 19, 29	$W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$

The **remaining cases** in the root system induction construction work the same way, not just this Trinity! So more **general correspondence**:

$$(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3) \rightarrow (I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$$

Aside for Adriana

rank 4	exponents	W-factorisation
A_4	1, 2, 3, 4	$W = \exp\left(\frac{\pi}{5} B_C\right) \exp\left(\frac{2\pi}{5} I B_C\right)$
B_4	1, 3, 5, 7	$W = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} I B_C\right)$
D_4	1, 3, 3, 5	$W = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{\pi}{2} I B_C\right)$
F_4	1, 5, 7, 11	$W = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} I B_C\right)$
H_4	1, 11, 19, 29	$W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$

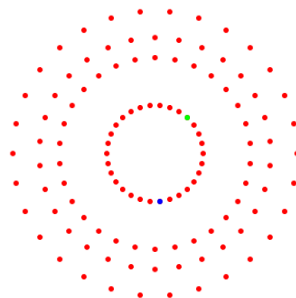
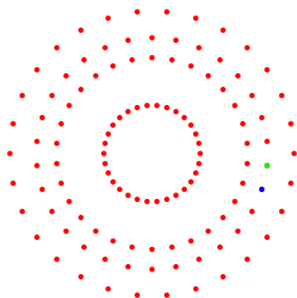
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4D case: H_4

- E.g. H_4 has exponents 1, 11, 19, 29
- Coxeter versor decomposes into **orthogonal components**

$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$$



2D/3D, 2D/4D and ADE correspondences

- McKay correspondence relates even $SU(2)$ subgroups with ADE Lie algebras ($A_{2n-1}, D_{n+2}, E_6, E_7, E_8$)
- Induction theorem: get these as 2D/4D root systems ($I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4$) from 2D/3D root systems ($I_2(n), A_1 \times I_2(n), A_3, B_3, H_3$)
- $(2n, 2n+2, 12, 18, 30)$ are numbers of roots, the sum of the dimensions of the irreps and the ADE Coxeter number

	4D	G	$\sum d_i$	ADE	h
				\tilde{A}_{2n-1}	$2n$
	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$	\tilde{D}_{n+2}	$2(n+1)$
	D_4	$2T$	12	\tilde{E}_6	12
	F_4	$2O$	18	\tilde{E}_7	18
	H_4	$2I$	30	\tilde{E}_8	30

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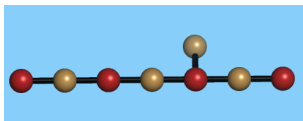
2D/3D	$ \Phi $	4D	G	$\sum d_i$	ADE	h
$A_1 \times I_2(n)$	$2n+2$	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$		
A_3	12	D_4	$2T$	12		
B_3	18	F_4	$2O$	18		
H_3	30	H_4	$2I$	30		

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- $(2n, 2n+2, 12, 18, 30)$ are **numbers of roots**, the sum of the **dimensions of the irreps** and the **ADE Coxeter number**

2D/3D	$ \Phi $	4D	G	$\sum d_i$	ADE	h
$I_2(n)$	$2n$	$I_2(n)$	C_{2n}	$2n$	\tilde{A}_{2n-1}	$2n$
$A_1 \times I_2(n)$	$2n+2$	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$	\tilde{D}_{n+2}	$2(n+1)$
A_3	12	D_4	$2T$	12	\tilde{E}_6	12
B_3	18	F_4	$2O$	18	\tilde{E}_7	18
H_3	30	H_4	$2I$	30	\tilde{E}_8	30

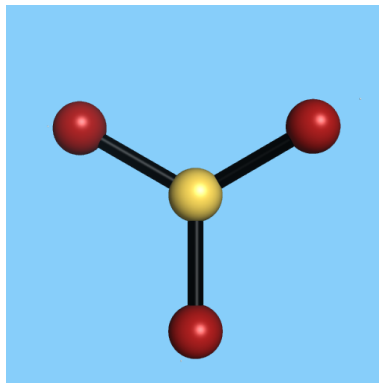
Is there a direct Platonic-ADE correspondence?



2D/3D		rot	ADE		legs
$I_2(n)$		n	A_n		n
$A_1 \times I_2(n)$		$2, 2, n$	D_{n+2}		$2, 2, n$
A_3		$2, 3, 3$	E_6		$2, 3, 3$
B_3		$2, 3, 4$	E_7		$2, 3, 4$
H_3		$2, 3, 5$	E_8		$2, 3, 5$

A Trinity of root system ADE correspondences

- **2D/3D** root systems $(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3)$
- **2D/4D** root systems $(I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$
- **ADE** root systems $(A_n, D_{n+2}, E_6, E_7, E_8)$



Outlook for algebraic aspects

- All exceptional geometries arise in 3D in a novel Clifford spinorial way, root systems giving rise to Lie groups/algebras
- New view of Coxeter plane geometry: degrees and exponents with geometric interpretation of imaginaries
- A unified framework for doing group and representation theory: polyhedral, orthogonal, conformal, modular (Moonshine) etc
- Conceptual unification at the level of root systems
- ADE correspondences between 2D/3D, 2D/4D & ADE root systems

Thank you!

Clifford Algebra and orthogonal transformations

- **Inner product** is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting a in b is given by $a' = a - 2(a \cdot b)b = -bab$ (b and $-b$ **doubly cover** the same reflection)
- Via **Cartan-Dieudonné** theorem any orthogonal (/conformal/modular) transformation can be written as **successive reflections**

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1 = \pm A x \tilde{A}$$

- The conformal group $C(p, q) \sim SO(p+1, q+1)$ so can use these for **translations, inversions** etc as well

Conformal Geometric Algebra

- Go to e_1, e_2, e, \bar{e} , with $e_0^2 = 1, e_i^2 = -1, e^2 = 1, \bar{e}^2 = -1$
- Define two **null** vectors $n \equiv e + \bar{e}, \bar{n} \equiv e - \bar{e}$
- Can **embed** the 2D vector $x = x^\mu e_\mu = xe_1 + ye_2$ as a **null vector in 4D** (also normalise $\hat{X} \cdot e = -1$)

$$\hat{X} = \frac{1}{\lambda^2 - x^2} (x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

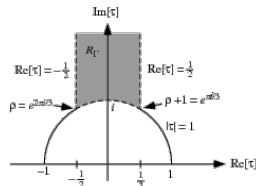
- So neat thing is that **conformal transformations** are now done by **rotors** (except inversion which is a reflection) – distances are given by **inner products**

Conformal Transformations in CGA

$$F(x) = \frac{1}{\lambda^2 - x^2} (x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

- **Reflection:** spacetime $F(-axa) = -aF(x)a$
- **Rotation:** spacetime $F(Rx\tilde{R}) = RF(x)\tilde{R}$, $R = \exp(\frac{ab}{2\lambda})$
- **Translation:** $F(x+a) = R_T F(x)\tilde{R}_T$ for $R_T = \exp(\frac{na}{2\lambda}) = 1 + \frac{na}{2\lambda}$
- **Dilation:** $F(e^\alpha x) = R_D F(x)\tilde{R}_D$ for $R_D = \exp(\frac{\alpha}{2\lambda} e\bar{e})$
- **Inversion:** Reflection in extra dimension e : $F(\frac{x}{x^2}) = -eF(x)e$
 sends $n \leftrightarrow \bar{n}$
- **Special conformal transformation:** $F(\frac{x}{1+ax}) = R_S F(x)\tilde{R}_S$ for
 $R_S = R_I R_T R_I$

Modular group



- Modular generators: $T : \tau \rightarrow \tau + 1$, $S : \tau \rightarrow -1/\tau$
- $\langle S, T | S^2 = I, (ST)^3 = I \rangle$ CGA: $R_Y X \tilde{R}_Y$
- CGA: $T_X = \exp\left(\frac{ne_1}{2}\right) = 1 + \frac{ne_1}{2}$ and $S_X = e_1 e$ (slight issue of complex structure $\tau =$ complex number, not vector in the 2D real plane so map $e_1 : x_1 e_1 + x_2 e_2 \leftrightarrow x_1 + x_2 e_1 e_2 = x_1 + ix_2$)
- $(S_X T_X)^3 = -1$ and $S_X^2 = -1$
- So a 3-fold and a 2-fold rotation in conformal space

Braid group

- $(S_X T_X)^3 = -1$ and $S_X^2 = -1$ is inherently **spinorial**
- Of course Clifford construction gives a **double cover**
- The **braid group** is a double cover
- So **Clifford** construction gives the **braid group double cover** of the **modular group**
- $\sigma_1 = \tilde{T}_X = \exp(-ne_1/2)$ and $\sigma_2 = T_X S_X T_X = \exp(-\bar{n}e_1/2)$ satisfying $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 (= S_X)$
- Nice **symmetry** between the roles of the **point at infinity** and the **origin**
- Might not be known? **Spinorial techniques** might make awkward **modular transformations** more tractable?