Root systems & Clifford algebras: from symmetries of viruses to E_8 & an ADE correspondence

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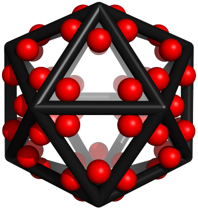
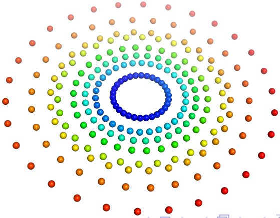
St Andrews Pure Maths Colloquium – January 13, 2017

Main results

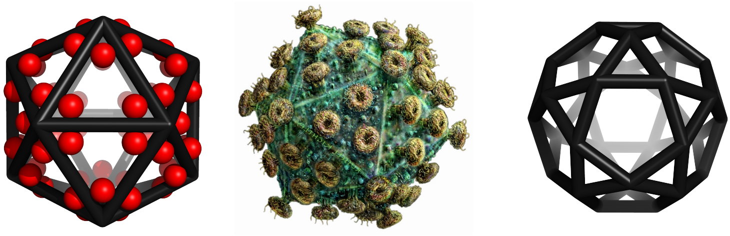
- New affine symmetry principle for viruses and fullerenes
- H_3 (icosahedral symmetry) induces the E_8 root system
- Each 3D root system induces a 4D root system
- This correspondence extends to exponents in the Coxeter plane (not just the original Trinity) and ADE Lie algebras
- Clifford algebra is a very natural framework for root systems and reflection groups in general





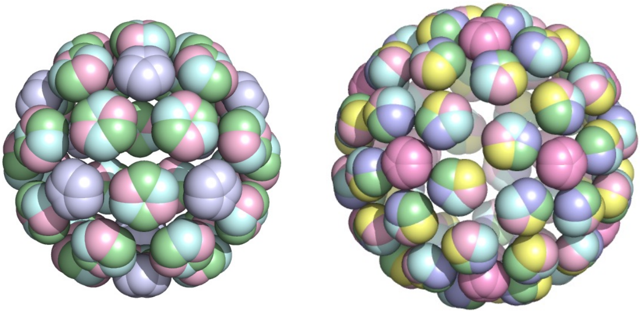
Overview

The Icosahedron



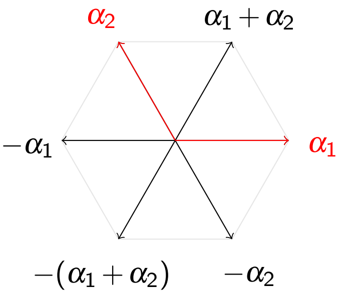
- Rotational icosahedral group is $I = A_5$ of order 60
- Full icosahedral group is the Coxeter group H_3 of order 120 (including reflections/inversion); generated by the root system icosidodecahedron

Icosahedral viruses



Two viral surface layouts: a $T = 4$ Caspar-Klug quasiequivalent triangulation, and a pseudo $T = 7$ defying Caspar-Klug theory, which is based on a kite-rhombus tiling instead (HPV)

Root systems



Root system Φ: set of vectors α in a vector space with an inner product such that

1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall \alpha \in \Phi$
2. $s_\alpha \Phi = \Phi \ \forall \alpha \in \Phi$

Simple roots: express every element of Φ via a $\mathbb{Z}$-linear combination.

reflection/Coxeter groups

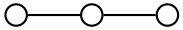
$$s_\alpha : v \rightarrow s_\alpha(v) = v - 2\frac{(v|\alpha)}{(\alpha|\alpha)}\alpha$$

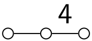
Cartan Matrices

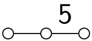
Cartan matrix of α_is is $\boxed{A_{ij} = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} = 2\frac{|\alpha_j|}{|\alpha_i|}\cos\theta_{ij}}$

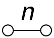
$$A_2\text{: } A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal, simple link = roots at $\frac{\pi}{3}$, link with label m = angle $\frac{\pi}{m}$.

A_3 

B_3 

H_3 

$I_2(n)$ 

Lie groups to Lie algebras to Coxeter groups to root systems

- Lie group: manifold of continuous symmetries (gauge theories, spacetime)
- Lie algebra: infinitesimal version near the identity
- Non-trivial part is given by a root lattice
- Weyl group is a crystallographic Coxeter group: $\boxed{A_n, B_n/C_n, D_n, G_2, F_4, E_6, E_7, E_8}$ generated by a root system.
- So via this route root systems are always crystallographic. Neglect non-crystallographic root systems $\boxed{I_2(n), H_3, H_4}$.

Affine extensions

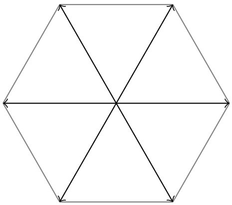
An affine Coxeter group is the extension of a Coxeter group by an affine reflection in a hyperplane not containing the origin $s_{\alpha_0}^{aff}$ whose geometric action is given by

$$\boxed{s_{\alpha_0}^{aff} v = \alpha_0 + v - \frac{2(\alpha_0|v)}{(\alpha_0|\alpha_0)}\alpha_0}$$

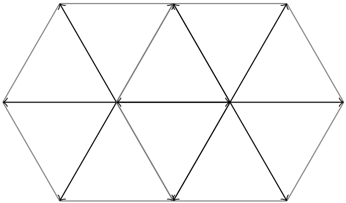
Non-distance preserving: includes the translation generator

$$\boxed{Tv = v + \alpha_0 = s_{\alpha_0}^{aff} s_{\alpha_0} v}$$

Affine extensions – A_2

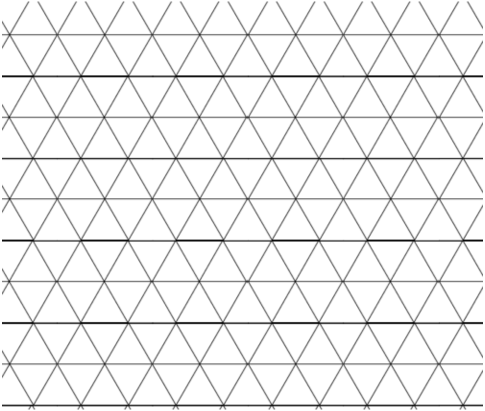


Affine extensions – A_2

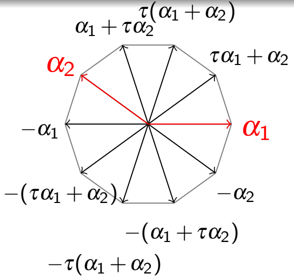


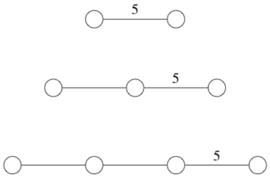
Affine extensions – A_2

Affine extensions of crystallographic Coxeter groups lead to a tessellation of the plane and a lattice.



Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$





$$A=\begin{pmatrix}2 & -\tau\\ -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -\tau\\ 0 & -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -\tau\\ 0 & 0 & -\tau & 2\end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate rotational symmetries of order 5

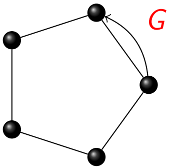
linear combinations now in the extended integer ring

$\mathbb{Z}[\tau]=\{a+\tau b | a,b\in\mathbb{Z}\}$ golden ratio $\tau=\frac{1}{2}(1+\sqrt{5})=2\cos\frac{\pi}{5}$

$x^2=x+1$ $\quad \tau'=\sigma=\frac{1}{2}(1-\sqrt{5})=2\cos\frac{2\pi}{5}$ $\quad \tau+\sigma=1, \tau\sigma=-1$

Affine extensions of non-crystallographic root systems?

Unit translation along a vertex of a unit pentagon



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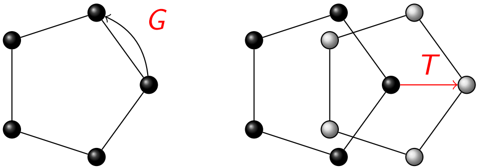

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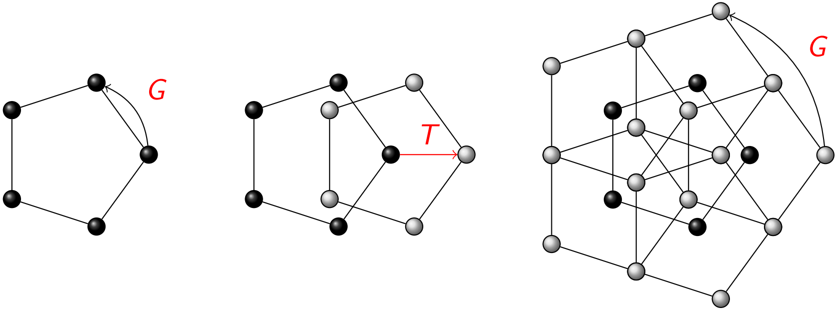
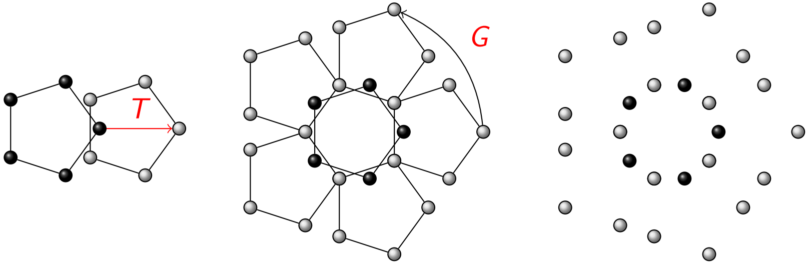


A random translation would give 5 secondary pentagons, i.e. 25 points. Here we have degeneracies due to 'coinciding points'.

Affine extensions of non-crystallographic root systems?

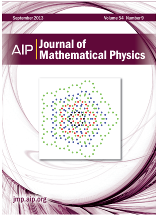
Translation of length $\tau = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$ (golden ratio)

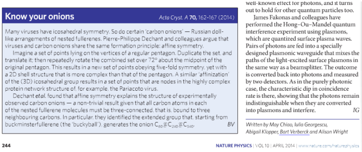


Cartoon version of a virus or carbon onion. Would there be an evolutionary benefit to have more than just compact symmetry? The problem has an intrinsic length scale.

Affine extensions of non-crystallographic Coxeter groups

- 2D and 3D point arrays for applications to viruses, fullerenes, quasicrystals, proteins etc
- Two complementary ways to construct these





Know your onions *Acta Cryst. A* **70,** 162-167 (2014)

Many viruses have icosahedral symmetry. So do certain 'carbon onions' — Russian doll-like arrangements of nested fullerenes. Pierre-Philippe Dechant and colleagues argue that viruses and carbon onions share the same formation principle: affine symmetry.

Imagine a set of points lying on the vertices of a regular pentagon. Duplicate the set, and translate it; then repeatedly rotate the combined set over 72° about the midpoint of the original pentagon. This results in a new set of points obeying five-fold symmetry, yet with a 2D shell structure that is more complex than that of the pentagon. A similar 'affinization' of the (3D) icosahedral group results in a set of points that are nodes in the highly complex protein network structure of, for example, the Pariacoto virus.

Dechant *et al.* found that affine symmetry explains the structure of experimentally observed carbon onions — a non-trivial result given that all carbon atoms in each of the nested fullerene molecules must be three-connected, that is, bound to three neighbouring carbons. In particular, they identified the extended group that, starting from buckminsterfullerene (the 'buckyball'), generates the onion $C_{60}@C_{240}@C_{540}$. *BV*

well-known effect for photons, and it turns out to hold for other quantum particles too.

James Fakonas and colleagues have performed the Hong–Ou–Mandel quantum interference experiment using plasmons, which are quantized surface plasma waves. Pairs of photons are fed into a specially designed plasmonic waveguide that mixes the paths of the light-excited surface plasmons in the same way as a beamsplitter. The outcome is converted back into photons and measured by two detectors. As in the purely photonic case, the characteristic dip in coincidence rate is there, showing that the photons remain indistinguishable when they are converted into plasmons and interfere. *IG*

Written by May Chiao, Iulia Georgescu, Abigail Klopper, Bart Verberck and Alison Wright

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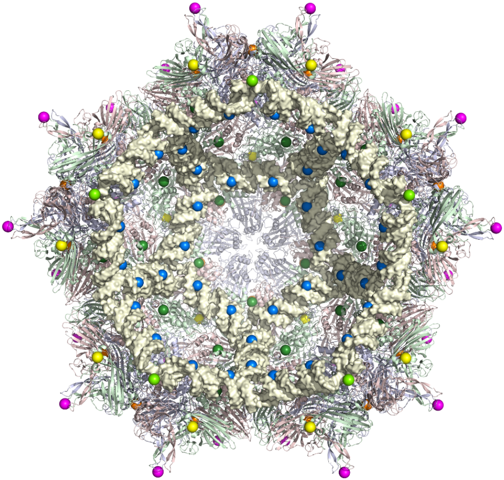
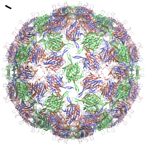
Use in Mathematical Virology



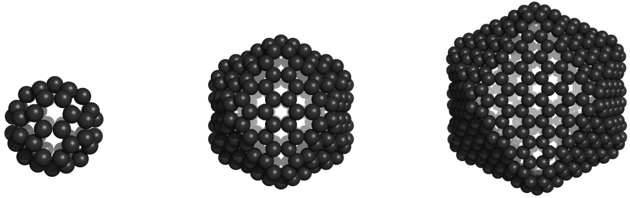
New insight into RNA virus assembly

- There are specific interactions between RNA and coat protein (CP) given by symmetry axes
- Essential for assembly as only this RNA-CP interaction turns CP into right geometric shape for capsid formation
- The RNA forms a Hamiltonian cycle visiting each CP once – dictated by symmetry
- A patent for a new antiviral strategy (Reidun Twarock)

Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{60} - C_{240} - C_{540}$)



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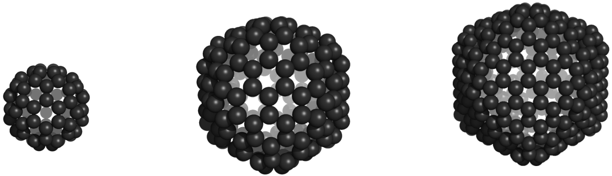
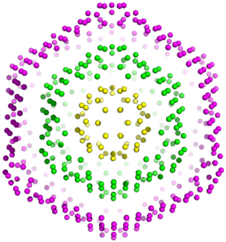
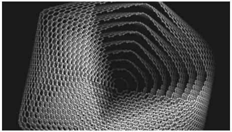
- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{80} - C_{180} - C_{320}$)



Viruses and fullerenes – symmetry as a common thread?

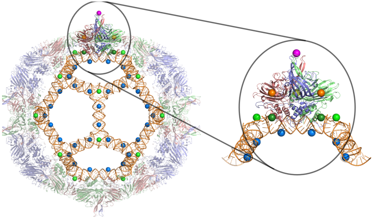
- Get nested arrangements like Russian dolls: carbon onions (e.g. Nature 510, 250253)
- Potential to extend to other known carbon onions with different start configuration, chirality etc

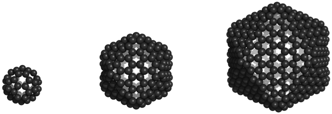




Two major areas for affine extensions of non-crystallographic Coxeter groups

- Non-compact symmetry that relates different structural features in the same polyhedral object
- Novel symmetry principle in Nature, shown that it seems to apply to at least fullerenes and viruses





Overview

Clifford Algebra and orthogonal transformations

- Form an algebra using the product between two vectors

$$\boxed{ab \equiv a \cdot b + a \wedge b}$$

- Inner product is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting x in n is given by $\boxed{x' = x - 2(x \cdot n)n = -nxn}$ (n and $-n$ doubly cover the same reflection)
- Via Cartan-Dieudonné theorem any orthogonal (/conformal/modular) transformation can be written as successive reflections

$$\boxed{x' = \pm n_1 n_2 \ldots n_k x n_k \ldots n_2 n_1} = \pm A x \tilde{A}$$

Clifford Algebra of 3D

- E.g. Pauli algebra in 3D (likewise for Dirac algebra in 4D) is

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

- We can multiply together root vectors in this algebra $\alpha_i\alpha_j\ldots$
- A general element has 8 components, even products (rotations/spinors) have four components:

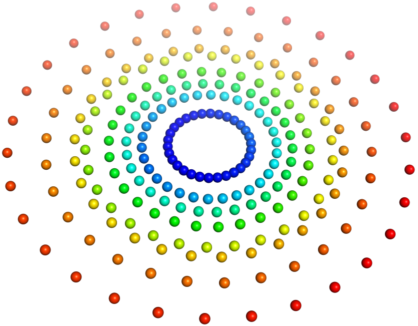
$$\boxed{R = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2 \Rightarrow R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

- So behaves as a 4D Euclidean object – inner product

$$\boxed{(R_1, R_2) = \frac{1}{2}(R_2\tilde{R}_1 + R_1\tilde{R}_2)}$$

Exceptional E_8 (projected into the Coxeter plane)

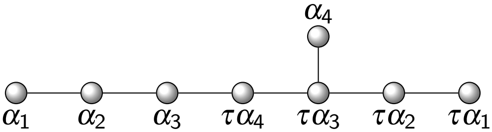
E_8 root system has 240 roots, H_3 has order 120

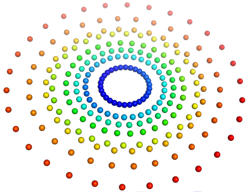


- Order 120 group H_3 doubly covered by 240 (s)pinors in 8D space

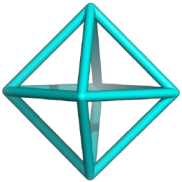
- With (somewhat counterintuitive) reduced inner product this gives the E_8 root system

- E_8 is actually hidden within 3D geometry!





Spinors from reflections



- The 6 roots $(\pm 1, 0, 0)$ and permutations of $A_1 \times A_1 \times A_1$ generate 8 spinors:
- $\boxed{\pm e_1,\ \pm e_2,\ \pm e_3}$ give the 8 spinors $\boxed{\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1}$
- This is a discrete spinor group isomorphic to the quaternion group Q.
- As 4D vectors these are $(\pm 1, 0, 0, 0)$ and permutations, the 8 roots of $A_1 \times A_1 \times A_1 \times A_1$ (the 16-cell).

Induction Theorem – root systems

- Induction Theorem: every 3D root system gives a 3D spinor group which gives a 4D root system.

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Induction Theorem – root systems

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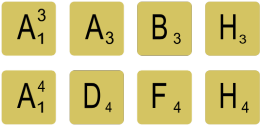
  1. $\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \;\forall\, \alpha \in \Phi}$
  2. $\boxed{s_\alpha \Phi = \Phi \;\forall\, \alpha \in \Phi}$
- Proof: 1. R and $-R$ are in a spinor group by construction (double cover of orthogonal transformations), 2. closure under reflections is guaranteed by the closure property of the spinor group (with a twist: $-R_1\tilde{R}_2R_1$)

H_4 from H_3

- The H_3 root system has 30 roots e.g. simple roots
$$\boxed{\alpha_1 = e_2, \alpha_2 = -\frac{1}{2}((\tau-1)e_1 + e_2 + \tau e_3) \text{ and } \alpha_3 = e_3}.$$
- Subgroup of rotations A_5 of order 60 is doubly covered by 120 spinors of the form $\boxed{\alpha_1\alpha_2 = -\frac{1}{2}(1-(\tau-1)e_1e_2 + \tau e_2e_3)}$,
$\boxed{\alpha_1\alpha_3 = e_2e_3}$ and $\boxed{\alpha_2\alpha_3 = -\frac{1}{2}(\tau-(\tau-1)e_3e_1 + e_2e_3)}$.
-
$$(\pm 1, 0, 0, 0) \text{ (8 perms) }, \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) \text{ (16 perms)}$$
$$\frac{1}{2}(0, \pm 1, \pm\sigma, \pm\tau) \text{ (96 even perms)},$$
As 4D vectors are the 120 roots of the H_4 root system.

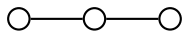
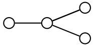
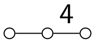
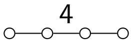
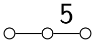
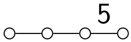
Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the Platonic Solids:
- The 6/12/18/30 roots in $A_1 \times A_1 \times A_1 / A_3 / B_3 / H_3$ generate 8/24/48/120 spinors.
- E.g. $\pm e_1$, $\pm e_2$, $\pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The discrete spinor group is isomorphic to the quaternion group Q / binary tetrahedral group $2T$/ binary octahedral group $2O$/ binary icosahedral group $2I$).



Exceptional Root Systems

- Exceptional phenomena: D_4 (triality, important in string theory), F_4 (largest lattice symmetry in 4D), H_4 (largest non-crystallographic symmetry); Exceptional D_4 and F_4 arise from series A_3 and B_3

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$	
A_3		$2T$	D_4	
B_3		$2O$	F_4	
H_3		$2I$	H_4	

Arnold's Trinities

Arnold's observation that many areas of real mathematics can be complexified and quaternionified resulting in theories with a similar structure.

- The fundamental trinity is thus $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The projective spaces $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The spheres $(\mathbb{R}P^1 = S^1, \mathbb{C}P^1 = S^2, \mathbb{H}P^1 = S^4)$
- The Möbius/Hopf bundles $(S^1 \to S^1, S^3 \to S^2, S^7 \to S^4)$
- The Lie Algebras (E_6, E_7, E_8)
- The symmetries of the Platonic Solids (A_3, B_3, H_3)
- The 4D groups (D_4, F_4, H_4)
- New connections via my Clifford spinor construction (see McKay correspondence)

Platonic Trinities

- Arnold's connection between (A_3, B_3, H_3) and (D_4, F_4, H_4) is very convoluted and involves numerous other trinities at intermediate steps:
- Decomposition of the projective plane into Weyl chambers and Springer cones
- The number of Weyl chambers in each segment is $24 = 2(1+3+3+5), 48 = 2(1+5+7+11), 120 = 2(1+11+19+29)$
- Notice this miraculously is one less than the degrees of invariants $((2,4,4,6),(2,6,8,12),(2,12,20,30))$ of the Coxeter groups (D_4, F_4, H_4)
- Believe the Clifford connection is more direct

A unified framework for polyhedral groups

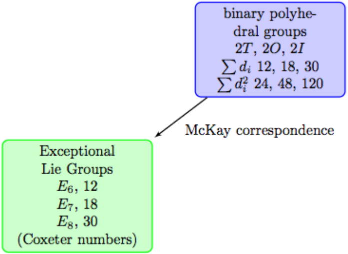
Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \to \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \to \pm\tilde{A}xA$
$\mathrm{Spin}(3)$	binary	$(R_1, R_2) \to R_1R_2$
$\mathrm{Pin}(3)$	pinor	$(A_1, A_2) \to A_1A_2$

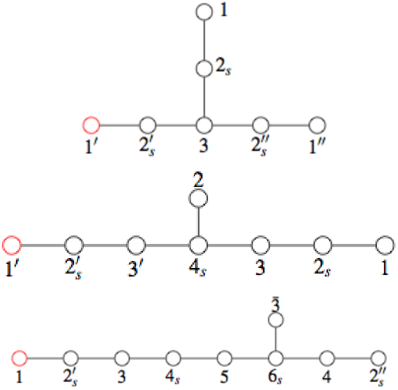
- e.g. the chiral icosahedral group has 60 elements, encoded in Clifford by 120 spinors, which form the binary icosahedral group
- together with the inversion/pseudoscalar I this gives 60 rotations and 60 rotoinversions, i.e. the full icosahedral group H_3 in 120 elements (with 240 pinors)

Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate conjugacy classes etc of pinors in Clifford algebra
- Chiral (binary) polyhedral groups have irreps
- tetrahedral (12/24): $1, 1', 1'', 2_s, 2_s', 2_s'', 3$
- octahedral (24/48): $1, 1', 2, 2_s, 2_s', 3, 3', 4_s$
- icosahedral (60/120): $1, 2_s, 2_s', 3, \bar{3}, 4, 4_s, 5, 6_s$
- Binary groups are discrete subgroups of $SU(2)$ and all thus have a 2_s spinor irrep
- Connection with the McKay correspondence!

The McKay Correspondence: Coxeter number, dimensions of irreps and tensor product graphs



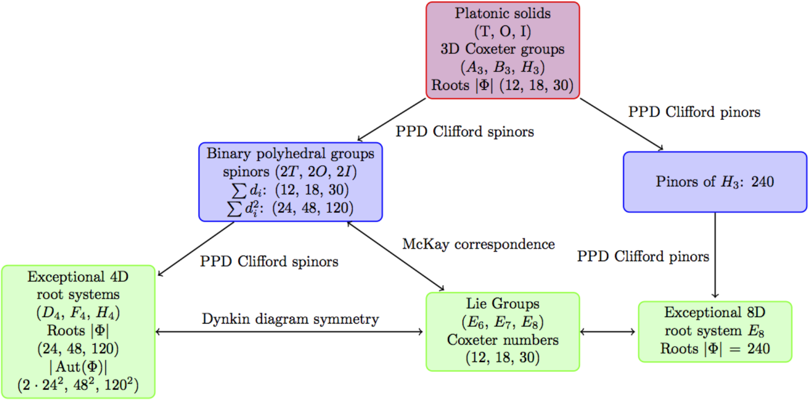


The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the binary cyclic and dicyclic groups are in correspondence with A_n and D_n, e.g. the quaternion group Q and D_4^+. So McKay correspondence not just a trinity but ADE-classification. We also have $I_2(n)$ on top of the trinity (A_3, B_3, H_3)

rank-3 group	diagram	binary	rank-4 group	diagram	Lie algebra	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$		D_4^+	
A_3		$2T$	D_4		E_6^+	
B_3	4	$2O$	F_4	4	E_7^+	
H_3	5	$2I$	H_4	5	E_8^+	

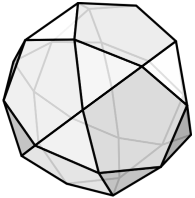
New explicit connections

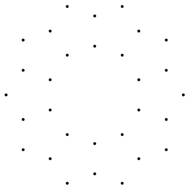


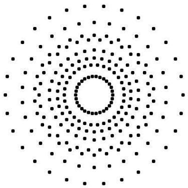
Overview

The Coxeter Plane

- Every (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by projecting their root system onto the Coxeter plane







Coxeter Elements, Degrees and Exponents

- Like the symmetric group, Coxeter groups can have invariant polynomials. Their degrees d are important invariants/group characteristics.
- Turns out that actually degrees d are intimately related to so-called exponents m $\boxed{m = d - 1}$.

Coxeter Elements, Degrees and Exponents

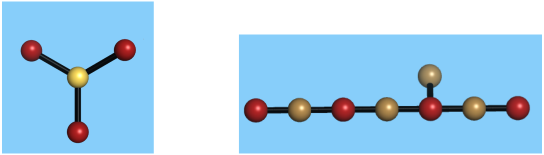
- A Coxeter Element is any combination of all the simple reflections $w = s_1 \dots s_n$, i.e. in Clifford algebra it is encoded by the versor $W = \alpha_1 \dots \alpha_n$ acting as $v \to wv = \pm \tilde{W} v W$. All such elements are conjugate and thus their order is invariant and called the Coxeter number h.
- The Coxeter element has complex eigenvalues of the form $\exp(2\pi mi/h)$ where m are called exponents:
  $wx = \exp(2\pi mi/h)x$
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again.

Coxeter Elements, Degrees and Exponents

- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where m are called exponents
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again.
- In particular, 1 and $h-1$ are always exponents
- Turns out that actually exponents and degrees are intimately related ($\boxed{m = d - 1}$). The construction is slightly roundabout but uniform, and uses the Coxeter plane.

The Coxeter Plane

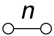
- In particular, can show every (for our purposes) Coxeter group has a Coxeter plane
- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of mutually commuting generators



The Coxeter Plane

- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of orthogonal = mutually commuting generators but anticommuting root vectors α_w and α_b (duals ω)
- Cartan matrices are positive definite, and thus have a Perron-Frobenius (all positive) eigenvector λ_i.
- Take linear combinations of components of this eigenvector as coefficients of two vectors from the orthogonal sets $v_w = \sum \lambda_w \omega_w$ and $v_b = \sum \lambda_b \omega_b$
- Their outer product/Coxeter plane bivector $\boxed{B_C = v_b \wedge v_w}$ describes an invariant plane where w acts by rotation by $2\pi/h$.

Clifford Algebra and the Coxeter Plane – 2D case

$I_2(n)$ 

- For $I_2(n)$ take $\boxed{\alpha_1 = e_1,\ \alpha_2 = -\cos\frac{\pi}{n}e_1 + \sin\frac{\pi}{n}e_2}$
- So Coxeter versor is just

$$\boxed{W = \alpha_1\alpha_2 = -\cos\frac{\pi}{n} + \sin\frac{\pi}{n}e_1e_2 = -\exp\left(-\frac{\pi I}{n}\right)}$$

- In Clifford algebra it is therefore immediately obvious that the action of the $I_2(n)$ Coxeter element is described by a versor (here a rotor/spinor) that encodes rotations in the e_1e_2-Coxeter-plane and yields $h = n$ since trivially $W^n = (-1)^{n+1}$ yielding $\boxed{w^n = 1}$ via $wv = \tilde{W}vW$.

Clifford Algebra and the Coxeter Plane – 2D case

- So Coxeter versor is just $\boxed{W = -\exp\left(-\frac{\pi I}{n}\right)}$
- $\boxed{I = e_1 e_2}$ anticommutes with both e_1 and e_2 such that sandwiching formula becomes $\boxed{v \to wv = \tilde{W} v W = \tilde{W}^2 v = \exp\left(\pm\frac{2\pi I}{n}\right) v}$ immediately yielding the standard result for the complex eigenvalues in real Clifford algebra without any need for artificial complexification
- The Coxeter plane bivector $B_C = e_1 e_2 = I$ gives the complex structure
- The Coxeter plane bivector B_C is invariant under the Coxeter versor $\tilde{W} B_C W = \pm B_C$.

Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can factorise W into orthogonal (commuting/anticommuting) components
  $W = \alpha_1 \dots \alpha_n = W_1 \dots W_n$ with $W_i = \exp(\pi m_i I_i / h)$
- Here, I_i is a bivector describing a plane with $I_i^2 = -1$
- For v orthogonal to the plane descrbed by I_i we have
  $v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i W_i v = v$ so cancels out
- For v in the plane we have
  $v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i / h) v$
- Thus if we decompose W into orthogonal eigenspaces, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

Clifford algebra: no need for complexification

- For v in the plane we have

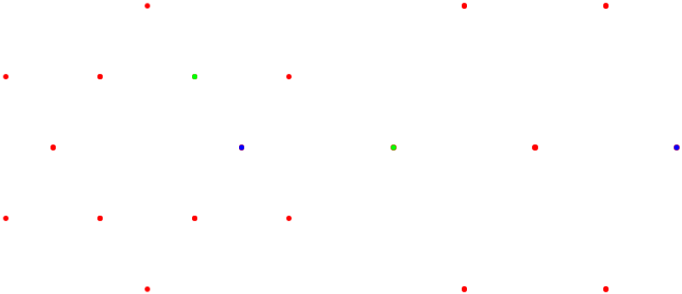
$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i / h) v$$

- So complex eigenvalue equation arises geometrically without any need for complexification
- Different complex structures immediately give different eigenplanes
- Eigenvalues/angles/exponents given from just factorising $W = \alpha_1 \dots \alpha_n$
- E.g. H_4 has exponents $1, 11, 19, 29$ and $W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$
- Here we have been looking for orthogonal eigenspaces, so innocuous – different complex structures commute
- But not in general – naive complexification can be misleading

4D case: D_4

- E.g. D_4 has exponents $1, 3, 3, 5$
- Coxeter versor decomposes into orthogonal components

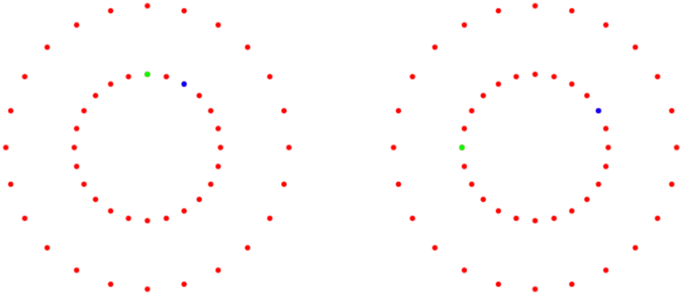
$$W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{6}B_C\right)\exp\left(\frac{3\pi}{6}IB_C\right)$$



4D case: F_4

- E.g. F_4 has exponents $1, 5, 7, 11$
- Coxeter versor decomposes into orthogonal components

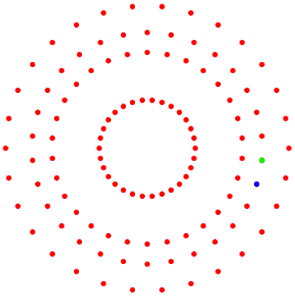
$$W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{12}B_C\right)\exp\left(\frac{5\pi}{12}IB_C\right)$$

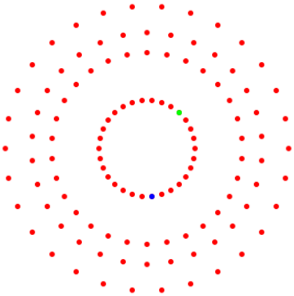


4D case: H_4

- E.g. H_4 has exponents $1, 11, 19, 29$
- Coxeter versor decomposes into orthogonal components

$$W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{11\pi}{30}IB_C\right)$$





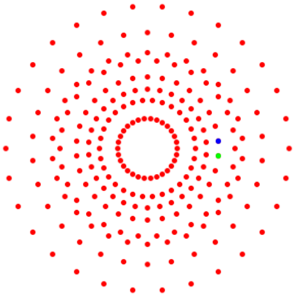
Clifford Algebra and the Coxeter Plane – 4D case summary

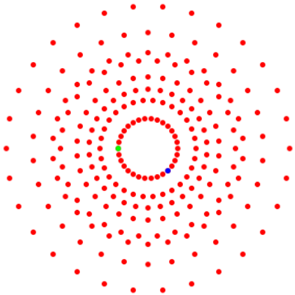
rank 4	exponents	W-factorisation
A_4	$1,2,3,4$	$W = \exp\left(\frac{\pi}{5}B_C\right)\exp\left(\frac{2\pi}{5}IB_C\right)$
B_4	$1,3,5,7$	$W = \exp\left(\frac{\pi}{8}B_C\right)\exp\left(\frac{3\pi}{8}IB_C\right)$
D_4	$1,3,3,5$	$W = \exp\left(\frac{\pi}{6}B_C\right)\exp\left(\frac{\pi}{2}IB_C\right)$
F_4	$1,5,7,11$	$W = \exp\left(\frac{\pi}{12}B_C\right)\exp\left(\frac{5\pi}{12}IB_C\right)$
H_4	$1,11,19,29$	$W = \exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{11\pi}{30}IB_C\right)$

8D case: E_8

- E.g. H_4 has exponents $1, 11, 19, 29$, E_8 has $1, 7, 11, 13, 17, 19, 23, 29$
- Coxeter versor decomposes into orthogonal components

$$W = \alpha_1 \dots \alpha_8 = \exp(\frac{\pi}{30} B_C) \exp(\frac{7\pi}{30} B_2) \exp(\frac{11\pi}{30} B_3) \exp(\frac{13\pi}{30} B_4)$$

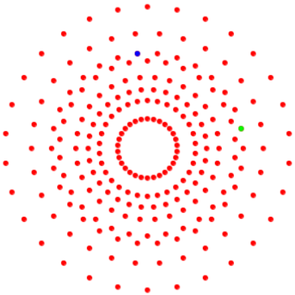


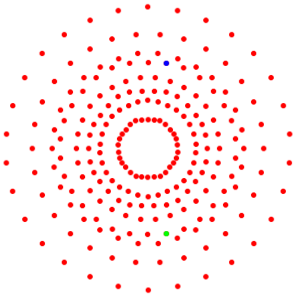


8D case: E_8

- E.g. H_4 has exponents $1, 11, 19, 29$, E_8 has $1, 7, 11, 13, 17, 19, 23, 29$
- Coxeter versor decomposes into orthogonal components

$$W = \alpha_1 \dots \alpha_8 = \exp(\frac{\pi}{30} B_C) \exp(\frac{7\pi}{30} B_2) \exp(\frac{11\pi}{30} B_3) \exp(\frac{13\pi}{30} B_4)$$





Imaginary differences – different imaginaries

So what has been gained by this Clifford view?

- There are different entities that serve as unit imaginaries
- They have a geometric interpretation as an eigenplane of the Coxeter element
- These don't need to commute with everything like i (though they do here – at least anticommute. But that is because we looked for orthogonal decompositions)
- But see that in general naive complexification can be a dangerous thing to do – unnecessary, issues of commutativity, confusing different imaginaries etc

The countably infinite family $I_2(n)$ and ADE

- For A_1^3 can see immediately $8 = 2(1+1+1+1)$
- Simple roots $\alpha_1 = e_1$, $\alpha_2 = e_2$, $\alpha_3 = e_3$, $\alpha_4 = e_4$ give $W = e_1e_2e_3e_4 = (\cos\frac{\pi}{2} + \sin\frac{\pi}{2}e_1e_2)(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}e_3e_4) = \exp(\frac{\pi}{2}e_1e_2)\exp(\frac{\pi}{2}e_3e_4)$
- Gives exponents $(1,1,1,1)$ (from $h-1 = 2-1$)

The countably infinite family $I_2(n)$ and ADE

- For $A_1 \times I_2(n)$ one gets the same decomposition $4n = 2(1+(n-1)+1+(n-1)) = 2\cdot 2n$
- Simple roots $\alpha_1 = e_1$, $\alpha_2 = -\cos\frac{\pi}{n}e_1 + \sin\frac{\pi}{n}e_2$, $\alpha_3 = e_3$, $\alpha_4 = -\cos\frac{\pi}{n}e_3 + \sin\frac{\pi}{n}e_4$ give $W = \exp\left(-\frac{\pi e_1 e_2}{n}\right)\exp\left(-\frac{\pi e_3 e_4}{n}\right)$
- Gives exponents $(1,(n-1),1,(n-1))$

The countably infinite family $I_2(n)$ and ADE

- So Arnold's initial hunch regarding the exponents extends in fact to my full correspondence
- McKay correspondence is a correspondence between even subgroups of $SU(2)$/quaternions and ADE affine Lie algebras
- In fact here get the even quaternion subgroups from 3D – link to ADE affine Lie algebras via McKay?

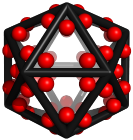
The countably infinite family $I_2(n)$, finite subgroups of $SU(2)$ and ADE

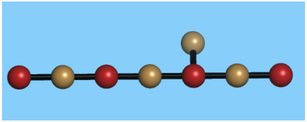
- Double cover of cyclic group of order n (binary cyclic group is $I_2(n)$ as a root system – corresponding to A_n affine Lie algebras (2D root systems are self-dual – just see complex part)
- The dicyclic group/binary dihedral group is $I_2(n) \times I_2(n)$ – corresponding to D_n affine Lie algebras
- The binary tetrahedral, octahedral and icosahedral groups are D_4, F_4 and H_4 as root systems – and correspond to E-type Lie algebras E_6, E_7, E_8
- Get all of the even subgroups of the quaternions from 2D and 3D root systems and via McKay correspondence the ADE Lie algebras

2D/3D, 2D/4D and ADE correspondences

- $(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3)$ give 4D root systems $(I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$
- These are in McKay correspondence with ADE Lie algebras $(A_{2n-1}, D_{n+2}, E_6, E_7, E_8)$
- The numbers of roots $(2n, 2n+2, 12, 18, 30)$ is also the sum of the dimensions of the irreps and the ADE Coxeter number
- Is there a direct correspondence between the Platonic root systems and the ADE root systems?

Yes there is

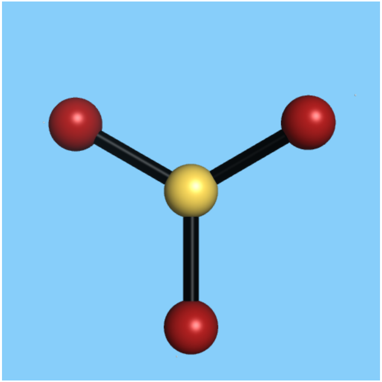




2D/3D		rot	ADE		legs
$I_2(n)$	n	n	A_n	$\cdots$	n
$A_1 \times I_2(n)$	n	$2, 2, n$	D_{n+2}	$\cdots$	$2, 2, n$
A_3		$2, 3, 3$	E_6		$2, 3, 3$
B_3	4	$2, 3, 4$	E_7		$2, 3, 4$
H_3	5	$2, 3, 5$	E_8		$2, 3, 5$

A Trinity of root system ADE correspondences

- 2D/3D root systems $(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3)$
- 2D/4D root systems $(I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$
- ADE root systems $(A_n, D_{n+2}, E_6, E_7, E_8)$

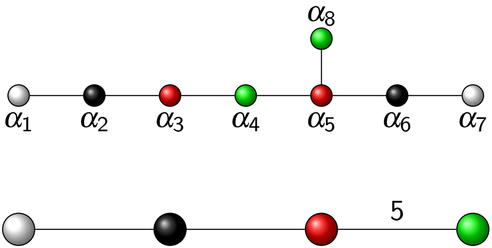


Conclusions

- **All** exceptional geometries arise in 3D, root systems giving rise to Lie groups/algebras etc
- Completely novel spinorial way of viewing exceptional geometries as 3D phenomena – implications for HEP etc?
- New view of Coxeter degrees and exponents with geometric interpretation of imaginaries
- A unified framework for doing group and representation theory: polyhedral, orthogonal, conformal, modular (Moonshine) etc
- Correspondence between 2D/3D & 2D/4D root systems extends to full ADE correspondence with McKay and generalising Arnold's link between trinities via 3D Weyl chamber decomposition and exponents in the 4D Coxeter plane

Thank you!

Projection and Diagram Foldings



$$s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}, \; s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}, \; s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}, \; s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4$$

- E_8 has a H_4 subgroup of rotations via a 'partial folding'
- Can project 240 E_8 roots to $H_4 + \tau H_4$ – essentially the reverse of the previous construction!
- Coxeter element & number of E_8 and H_4 are the same

4D geometry is surprisingly important for HEP

- 4D root systems are surprisingly relevant to HEP
- A_4 is $SU(5)$ and comes up in Grand Unification
- D_4 is $SO(8)$ and is the little group of String theory
- In particular, its triality symmetry is crucial for showing the equivalence of RNS and GS strings
- B_4 is $SO(9)$ and is the little group of M-Theory
- F_4 is the largest crystallographic symmetry in 4D and H_4 is the largest non-crystallographic group
- The above are subgroups of the latter two
- Spinorial nature of the root systems could have surprising consequences for HEP