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Dechant, Pierre-Philippe ORCID

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What Clifford algebra can do for Coxeter groups and root systems

Pierre-Philippe Dechant

CPT, Mathematics Department, Durham University

Durham Mathematics HEP seminar – February 21, 2014

The general theme: Geometry & Symmetry and their Applications

- Worked on a few different things: HEP – strings, particles and cosmology, pure maths and mathematical biology and Clifford algebras and mathematical physics
- Unifying themes of symmetry and geometry (euclidean, conformal, hyperbolic, spherical)
- Continuous Lie groups, e.g. for modeling cosmological spacetimes (Bianchi models)
- Discrete Coxeter groups and Kac-Moody algebras describe gravitational singularities/hidden symmetries in HEP theory, viruses, fullerenes, &c
- Mathematical frameworks of Coxeter groups and Clifford algebras

1 Introduction

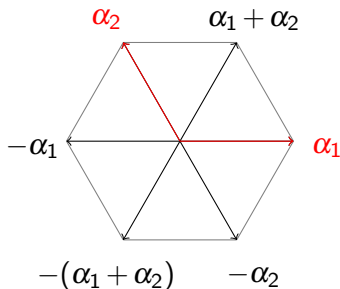
- Coxeter groups and root systems
- Clifford algebras

2 Coxeter and Clifford

- The Induction Theorem – from 3D to 4D
- The Coxeter Plane
- Conformal Geometry
- Some Group Theory

3 Moonshine and Outlook

Root systems – A_2

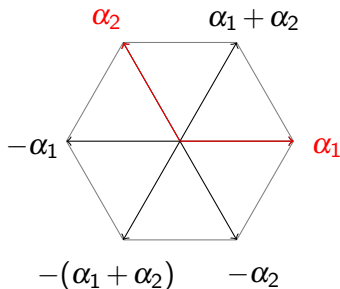


Root system Φ : set of
vectors α such that

$$\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$$

and $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

Root systems – A_2



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and $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

Simple roots: express every element of Φ via a **\mathbb{Z} -linear combination** (with coefficients of the same sign).

Cartan Matrices

Cartan matrix of α_i s is
$$A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2 \frac{|\alpha_j|}{|\alpha_i|} \cos \theta_{ij}$$

Cartan Matrices

Cartan matrix of α_i s is $A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2 \frac{|\alpha_j|}{|\alpha_i|} \cos \theta_{ij}$

$$\cos^2 \theta_{ij} = \frac{1}{4} A_{ij} A_{ji}$$

$$l_j^2 = \frac{A_{ij}}{A_{ji}} l_i^2$$

$$A_{ii} = 2$$

$$A_{ij} \in \mathbb{Z}^{\leq 0}$$

$$A_{ij} = 0 \Leftrightarrow A_{ji} = 0.$$

$$A_2: A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Cartan Matrices

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Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal, simple link = roots at $\frac{\pi}{3}$, link with label m = angle $\frac{\pi}{m}$.

$$A_2 \circ \text{---} \circ$$

$$H_2 \circ \text{---}^5 \circ$$

$$I_2(n) \circ \text{---}^n \circ$$

Coxeter groups

A **Coxeter group** is a group generated by some **involutive generators** $s_i, s_j \in S$ subject to relations of the form $(s_i s_j)^{m_{ij}} = 1$ with $m_{ij} = m_{ji} \geq 2$ for $i \neq j$.

Coxeter groups

A **Coxeter group** is a group generated by some **involutive generators** $s_i, s_j \in S$ subject to relations of the form $(s_i s_j)^{m_{ij}} = 1$ with $m_{ij} = m_{ji} \geq 2$ for $i \neq j$.

The **finite** Coxeter groups have a **geometric representation** where the involutions are realised as **reflections** at **hyperplanes through the origin** in a Euclidean vector space \mathcal{E} . In particular, let $(\cdot|\cdot)$ denote the inner product in \mathcal{E} , and $v, \alpha \in \mathcal{E}$.

The **generator** s_α corresponds to the **reflection**

$$s_\alpha : v \rightarrow s_\alpha(v) = v - 2 \frac{(v|\alpha)}{(\alpha|\alpha)} \alpha$$

at a hyperplane perpendicular to the **root vector** α .

The action of the **Coxeter group** is to permute these **root vectors**.

Coxeter groups vs Lie groups vs Lie algebras vs root systems

- Lie group = **group** and **manifold**
- Lie algebra = **bilinear, antisymmetric bracket** and **Jacobi identity**
- Lie algebras are **infinitesimal** version of Lie group = near the identity
- Can be more comprehensive e.g. 2D conformal algebra vs 2D conformal group
- But finite group transformation laws can be easier than linearising
- 'Nice' Lie algebras have **triangular decomposition**:
$$\mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$$

Coxeter groups vs Lie groups vs Lie algebras vs root systems

- 'Nice' Lie algebras have **triangular decomposition**:
 $\mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$
- \mathcal{H} is the **Cartan subalgebra** (maximal commuting = quantum numbers)
- Creation and annihilation algebras \mathcal{N} form **root lattice**
- Symmetry group is called **Weyl group** and is a **crystallographic** Coxeter group: $A_n, B_n/C_n, D_n, G_2, F_4, E_6, E_7, E_8$
- So Coxeter groups in theoretical physics always crystallographic! **Neglect** $I_2(n), H_3, H_4$.
- Useful Lie algebras are **(semi-)simple** LA (determinant of Cartan matrix is positive), **affine** LA (determinant is 0), **Kac-Moody algebras**, Borcherd's algebras...

Kac-Moody algebras

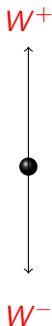
- Kac-Moody algebras \mathcal{A} of rank N are defined by **generalised Cartan** $(N \times N)$ matrices with $A_{ii} = 2$, $A_{ij} \in \mathbb{Z}_-(i \neq j)$ and $A_{ij} \neq 0 \Rightarrow A_{ji} \neq 0$
- $3N$ **generators** h_i, e_i, f_i satisfy **Chevalley-Serre relations**
 $[h_i, h_j] = 0$ $[h_i, e_j] = A_{ij}e_j$, $[h_i, f_j] = -A_{ij}f_j$, $[e_i, f_j] = \delta_{ij}h_i$
 $\underbrace{[e_i, [e_i, [\dots, [e_i, e_j]]]] \dots]}_{1-A_{ij} \text{ times}} = 0$, $\underbrace{[f_i, [f_i, [\dots, [f_i, f_j]]]] \dots]}_{1-A_{ij} \text{ times}} = 0$
- **Simple roots** α_i are $[h, e_i] = \alpha_i(h)e_i$

Example – A_1 , $SU(2)$, Angular Momentum



- Cartan subalgebra = Quantum number: L_z
- \mathcal{N}_+ : raising operator $L_+ = \alpha$
- \mathcal{N}_- : lowering operator $L_- = -\alpha$
- (L^2 is Casimir/commutes with all algebra elements, is however not actually in the algebra!)

Example – A_1 , $SU(2)$, Electroweak



- Cartan subalgebra – Quantum number: A
- \mathcal{N}_+ : raising operator $W^+ = \alpha$
- \mathcal{N}_- : lowering operator $W^- = -\alpha$
- (Since SM electroweak is actually $SU(2) \times U(1)$, $U(1)$ gives another field i , such that physical Z^0 and γ are superpositions of A and i)
- Also W^\pm now charged and self-interact, unlike QED

Affine extensions

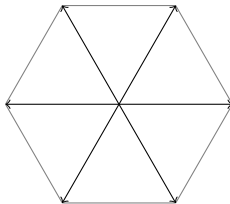
An **affine Coxeter group** is the extension of a Coxeter group by an **affine reflection in a hyperplane not containing the origin** $s_{\alpha_0}^{aff}$ whose geometric action is given by

$$s_{\alpha_0}^{aff} v = \alpha_0 + v - \frac{2(\alpha_0 | v)}{(\alpha_0 | \alpha_0)} \alpha_0$$

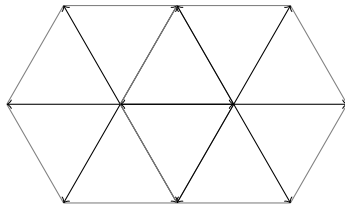
Non-distance preserving: includes the **translation generator**

$$T v = v + \alpha_0 = s_{\alpha_0}^{aff} s_{\alpha_0} v$$

Affine extensions – A_2

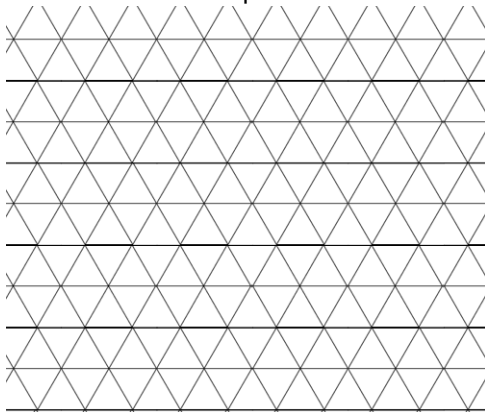


Affine extensions – A_2

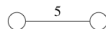
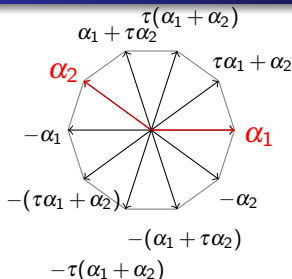


Affine extensions – A_2

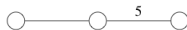
Affine extensions of crystallographic Coxeter groups lead to a **tessellation** of the plane and a **lattice**.



Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$



$$A = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate **rotational symmetries of order 5** linear combinations now in the **extended integer ring**

$$\mathbb{Z}[\tau] = \{a + \tau b \mid a, b \in \mathbb{Z}\} \quad \text{golden ratio}$$

$$\tau = \frac{1}{2}(1 + \sqrt{5}) = 2 \cos \frac{\pi}{5}$$

$$x^2 = x + 1$$

$$\tau' = \sigma = \frac{1}{2}(1 - \sqrt{5}) = 2 \cos \frac{2\pi}{5}$$

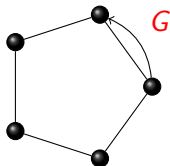
$$\tau + \sigma = 1, \tau\sigma = -1$$

What's new?

- In HEP, mostly come from Lie groups, then Lie algebras, then their Weyl groups and root systems
- This only gives the crystallographic Coxeter groups
- Do the non-crystallographic Coxeter groups have something interesting to offer? In particular, affine extensions?
- Interesting connections between the geometries of different dimensions: Relation between crystallographic and non-crystallographic (E_8 and H_4) and my spinor construction (3 & 4D)
- Both could have interesting consequences for HEP (4D groups and E_8 feature heavily) and other applications (viruses, quasicrystals, proteins, fullerenes...)

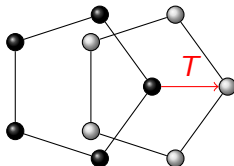
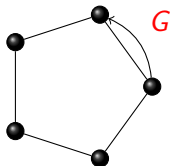
Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon



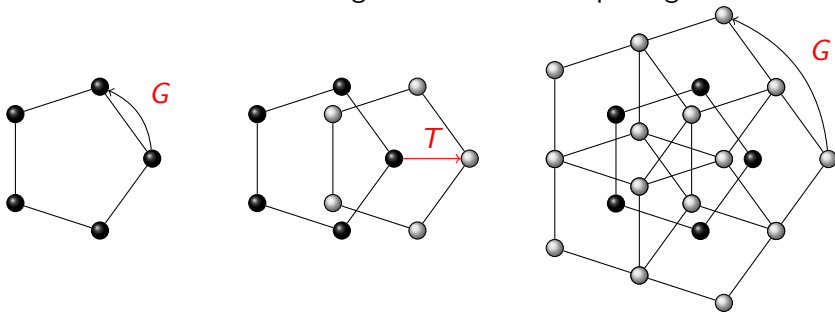
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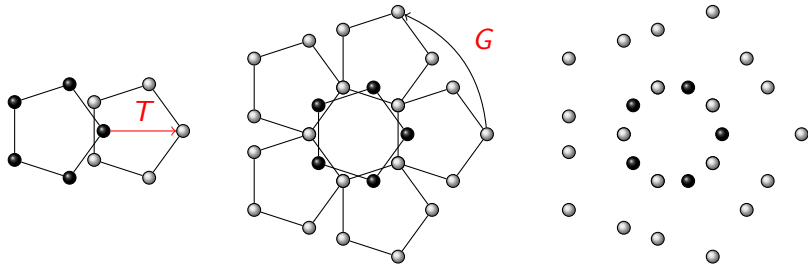
Unit translation along a vertex of a unit pentagon



A **random** translation would give 5 secondary pentagons, i.e. 25 points. Here we have **degeneracies** due to 'coinciding points'.

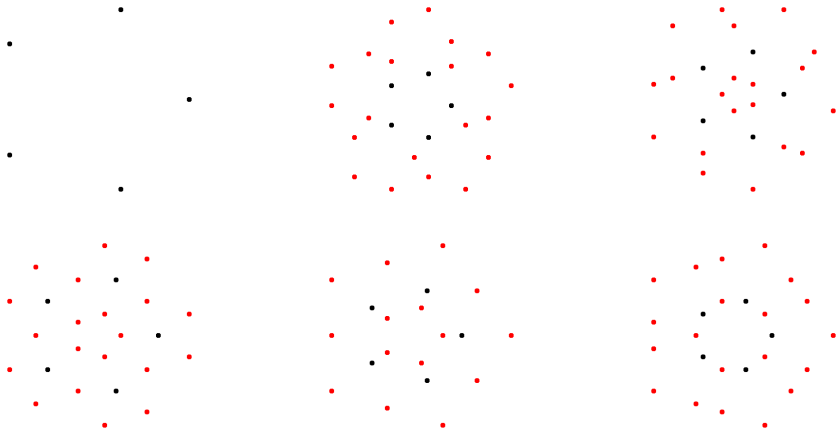
Affine extensions of non-crystallographic root systems

Translation of length $\tau = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ (golden ratio)

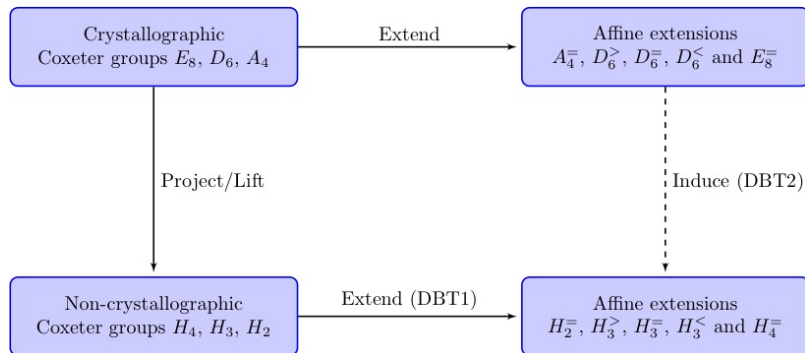


Looks like a **virus** or **carbon onion**

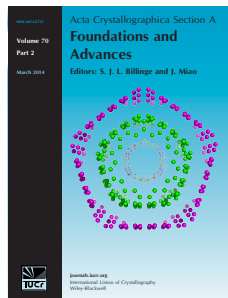
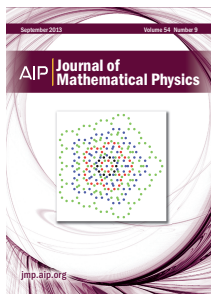
More Blueprints



Road Map



Applications of affine extensions of non-crystallographic root systems



There are interesting applications to **quasicrystals**, **viruses** or **carbon onions**, but here concentrate on the **mathematical** aspects

Basics of Clifford Algebra I

- Form an algebra using the **Geometric Product**

$$ab \equiv a \cdot b + a \wedge b \quad \text{for two vectors}$$

- Extend via linearity and associativity to higher grade elements (**multivectors**)
- For an n -dimensional space generated by n orthogonal unit vectors e_i have 2^n elements
- Then $e_i e_j = e_i \wedge e_j = -e_j e_i$ so **anticommute** (Grassmann variables, exterior algebra)
- Unlike the inner and outer products separately, this product is **invertible**
- This feeds through to the **differential structure** of the theory with more powerful **Greens functions methods** ∇^{-1}

Basics of Clifford Algebra II

- These are known to have **matrix representations** over the normed division algebras \mathbb{R} , \mathbb{C} and $\mathbb{H} \Rightarrow$ **Classification** of Clifford algebras
- E.g. **Pauli algebra** in 3D (likewise for **Dirac algebra** in 4D) is

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1 e_2 e_3\}}_{1 \text{ trivector}}$$

- These have the well-known matrix representations in terms of **σ - and γ -matrices**
- Working with these is not necessarily the most insightful thing to do, so here stress approach to **work directly** with the algebra
- Naturally have things that **square to -1** , e.g.

$$\boxed{(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1}, \text{ and } \text{non-trivial commutation properties}$$

Reflections

- Clifford algebra is **very efficient** at performing **reflections**
- Consider reflecting the vector a in a hypersurface with unit normal n :

$$a' = a_{\perp} - a_{\parallel} = a - 2a_{\parallel} = a - 2(a \cdot n)n$$

- c.f. **fundamental Weyl reflection** $s_i : v \rightarrow s_i(v) = v - 2 \frac{(v|\alpha_i)}{(\alpha_i|\alpha_i)} \alpha_i$
- But in Clifford algebra have $n \cdot a = \frac{1}{2}(na + an)$ so reassembles into **sandwiching**

$$a' = -nan$$

- So both **Coxeter** and **Clifford** frameworks are ideally suited to describing **reflections** – first to combine the two

Reflections and Rotations

- Generate a **rotation** when compounding two reflections wrt n then m (**Cartan-Dieudonné theorem**):

$$a'' = mn a n m \equiv R a \tilde{R}$$

where $R = mn$ is called a **rotor** and a tilde denotes **reversal** of the order of the constituent vectors ($R \tilde{R} = 1$)

- Now neat thing is all multivectors transform **covariantly** e.g.

$$MN \rightarrow (R M \tilde{R})(R N \tilde{R}) = R M \tilde{R} R N \tilde{R} = R(MN) \tilde{R}$$

so transform **double-sidedly**

- Rotors form a **group**, the rotor group, which gives a representation of the **Spin group** $Spin(n)$ – they transform **single-sidedly** (obvious now it's a double (universal) cover)

Artin's Theorem and orthogonal transformations

- **Artin**: every isometry is at most d reflections
- Since have a **double cover** of reflections (n and $-n$) we have a **double cover** of $O(p, q)$: **$\text{Pin}(p, q)$**

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1$$

- Pinors/vectors = products of vectors $n_1 n_2 \dots n_k$ encode orthogonal transformations via '**sandwiching**'
- **Cartan-Dieudonné**: rotations are an even number of reflections: **$\text{Spin}(p, q)$** doubly covers $SO(p, q)$
- The conformal group **$C(p, q) \sim SO(p+1, q+1)$** so can use these for **translations, inversions** etc as well

Spinor techniques

- Of course there is a **matrix representation** \underline{R} for the action of a **spinor**: $\underline{R}x = Rx\tilde{R}$
- This is the usual **rotation matrix** \underline{R} in $SO(p, q)$
- Having the **spin double cover/square root of the rotation matrix** can be convenient
- E.g. can get **differential equations for spinor** R that are **easier** to solve, then can reconstitute \underline{R} if necessary

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3D Platonic Solids



- There are 5 Platonic solids
- Tetrahedron (**self-dual**) (A_3)
- **Dual** pair **octahedron** and **cube** (B_3)
- **Dual** pair **icosahedron** and **dodecahedron** (H_3)
- Only the **octahedron** is a **root system** (actually for (A_1^3))

Clifford and Coxeter: Platonic Solids



Platonic Solid	Group	root system
Tetrahedron	A_3 A_1^3	Cuboctahedron Octahedron
Octahedron Cube	B_3	Cuboctahedron + Octahedron
Icosahedron Dodecahedron	H_3	Icosidodecahedron

- **Platonic Solids** have been known for millennia

Clifford and Coxeter: Platonic Solids



A_1^3

A_3

B_3

H_3

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- **Platonic Solids** have been known for millennia
- Described by **Coxeter** groups

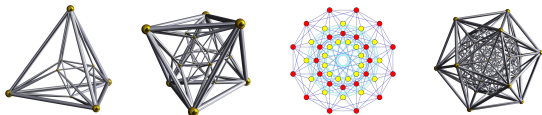
Clifford and Coxeter: Platonic Solids



A_1^3	A_1^4
A_3	D_4
B_3	F_4
H_3	H_4

- **Platonic Solids** have been known for millennia; described by **Coxeter** groups
- Concatenating reflections gives **Clifford** spinors (**binary polyhedral groups**)
- These **induce 4D root systems**

$$\psi = a_0 + a_i |e_i| \Rightarrow \psi \tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$
- 4D analogues of the Platonic Solids and give rise to 4D **Coxeter** groups



4D 'Platonic Solids'

- In 4D, there are **6 analogues** of the Platonic Solids:
- **5-cell** (self-dual) (A_4)
- **24-cell** (self-dual) (D_4) – a 24-cell and its dual together are the **F_4 root system**
- Dual pair **16-cell** and **8-cell** (B_4)
- Dual pair **600-cell** and **120-cell** (H_4)
- **24-cell, 16-cell and 600-cell** are all **root systems**, as is the related **F_4 root system**
- 8-cell and 120-cell are dual to a root system, so in 4D **out of 6 Platonic Solids only the 5-cell** (corresponding to A_n family) is not related to a root system!
- The 4D Platonic solids are **not normally thought to be related to the 3D ones** except for the boundary cells

Spinorial Symmetries of 4D Polytopes

Spinorial symmetries

rank 3	$ \Phi $	$ W $	rank 4	$ \Phi $	Symmetry
A_3	12	24	D_4 24-cell	24	$2 \cdot 24^2 = 576$
B_3	18	48	F_4 lattice	48	$48^2 = 2304$
H_3	30	120	H_4 600-cell	120	$120^2 = 14400$
A_1^3	6	8	A_1^4 16-cell	8	$3! \cdot 8^2 = 384$
$A_1 \oplus A_2$	8	12	$A_2 \oplus A_2$ prism	12	$12^2 = 144$
$A_1 \oplus H_2$	12	20	$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$A_1 \oplus I_2(n)$	$n+2$	$2n$	$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

Similar for **Grand Antiprism** (H_4 without $H_2 \oplus H_2$) and **Snub 24-cell** ($2I$ without $2T$).

Induction Theorem

- Theorem: 3D spinor groups are root systems (R and $-R$ are in a spinor group by construction, and closure under reflections is guaranteed by the closure property of the spinor group)
- Induction Theorem: Every rank-3 root system induces a rank-4 root system.
- Counterexample: not every rank-4 root system is induced in this way
- Spinor group is trivially closed under conjugation, left and right multiplication. Results in non-trivial symmetries when viewed as a polytope/root system.
- Now explains symmetry of the polytopes/root system and thus the order of the rank-4 Coxeter group

Induction Theorem

- So induced **4D polytopes** are actually **root systems**.
- Clear why the **number of roots** $|\Phi|$ is equal to $|G|$, the **order of the spinor group**
- Theorem: The **automorphism group** of the induced root system contains **two factors** of the respective spinor group acting from the **left** and the **right**.
- Only **remaining cases** in 3D are $A_1 \oplus I_2(n)$, which give $I_2(n) \oplus I_2(n)$

General Case of Induction

Only **remaining case** is what happens for $A_1 \oplus I_2(n)$ - this gives a **doubling** $I_2(n) \oplus I_2(n)$

rank 3	rank 4
A_3	D_4
B_3	F_4
H_3	H_4
A_1^3	A_1^4
$A_1 \oplus A_2$	$A_2 \oplus A_2$
$A_1 \oplus H_2$	$H_2 \oplus H_2$
$A_1 \oplus I_2(n)$	$I_2(n) \oplus I_2(n)$

Spinorial Symmetries of 4D Polytopes

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$A_1 \oplus H_2$	12	20	$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$A_1 \oplus I_2(n)$	$n+2$	$2n$	$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

Similar for **Grand Antiprism** (H_4 without $H_2 \oplus H_2$) and **Snub 24-cell** ($2I$ without $2T$). Additional factors in the automorphism group come from **3D Dynkin diagram symmetries**!

Some non-Platonic examples of spinorial symmetries

- **Grand Antiprism**: the 100 vertices achieved by subtracting 20 vertices of $H_2 \oplus H_2$ from the 120 vertices of the H_4 root system 600-cell – two separate orbits of $H_2 \oplus H_2$
- This is a semi-regular polytope with automorphism symmetry $\text{Aut}(H_2 \oplus H_2)$ of order $400 = 20^2$
- Think of the $H_2 \oplus H_2$ as coming from the **doubling procedure**? (Likewise for $\text{Aut}(A_2 \oplus A_2)$ subgroup)
- **Snub 24-cell**: $2T$ is a subgroup of $2I$ so subtracting the 24 corresponding vertices of the 24-cell from the 600-cell, one gets a semiregular polytope with 96 vertices and automorphism group $2T \times 2T$ of order $576 = 24^2$.

Sub root systems

- The above spinor groups had spinor multiplication as the **group operation**
- But also closed under **twisted conjugation** – corresponds to **closure under reflections** (root system property)
- If we take **twisted conjugation** as the group operation instead, we can have various **subgroups**
- These are the remaining **4D root systems** e.g. A_4 or B_4

What's new?

- Novel **connection** between geometry of **3D and 4D**
- In fact, 3D seems more **fundamental** – contrary to the **usual perspective** of 3D subgroups of 4D groups
- **Spinorial symmetries**
- Clear why **spinor group** gives a root system and why **two factors** of the same group reappear in the **automorphism group**
- Novel **spinorial perspective** on 4D geometry
- **Accidentalness** of the spinor construction and **exceptional** 4D phenomena
- Connection with Arnold's **trinities**, the **McKay correspondence** and **Monstrous Moonshine**

Recap: Clifford algebra and reflections & rotations

- Clifford algebra is **very efficient** at performing **reflections** via **sandwiching**

$$a' = -nan$$

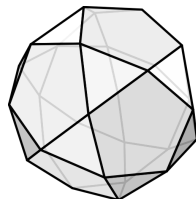
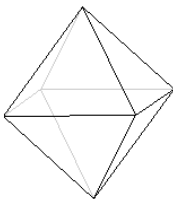
- Generate a **rotation** when compounding two reflections wrt n then m (**Cartan-Dieudonné theorem**):

$$a'' = mn an m \equiv R a \tilde{R}$$

where $R = mn$ is called a **rotor** and a tilde denotes **reversal** of the order of the constituent vectors ($R\tilde{R} = 1$)

From the Coxeter simple roots to the root system

- Take the $A_1 \times A_1 \times A_1$ simple roots $(1,0,0)$, $(0,1,0)$, $(0,0,1)$
 \Rightarrow under reflections get $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$, the
vertices of an **octahedron**.
- Take the **three simple roots** of $A_1 \times A_1 \times A_1 / A_3 / B_3 / H_3$.
Closure under Clifford **reflections** generate the whole root
system of 6/12/18/30 **vertices of an
octahedron/cuboctahedron/ cuboctahedron with an
octahedron/ icosidodecahedron**).



Spinors from reflections

- These are the 3D Coxeter groups that are symmetry groups of the **Platonic Solids** (tetrahedron and octahedron are similar but simpler than the icosahedron)
- The 6/12/18/30 **reflections** in $A_1 \times A_1 \times A_1 / A_3 / B_3 / H_3$ generate 8/24/48/120 **rotors**.
- E.g. $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ give the 8 permutations of $(\pm 1; 0, 0, 0)$ (scalar and bivector parts, the notation will become clear later).
- The **discrete spinor group** is isomorphic to the **quaternion group** Q / **binary tetrahedral group** $2T$ / **binary octahedral group** $2O$ / **binary icosahedral group** $2I$).

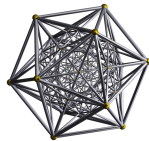
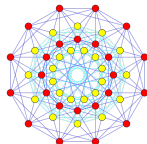
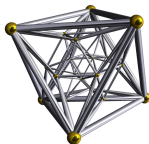
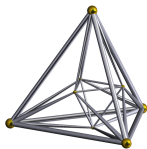
A unified framework for polyhedral groups

Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm \tilde{A}xA$
$Spin(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$Pin(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the **chiral icosahedral** group has 60 elements, encoded in Clifford by 120 rotors, which form the **binary icosahedral** group
- together with the **inversion/pseudoscalar** I this gives 60 rotations and 60 rotoinversions, i.e. the **full icosahedral** group H_3 in 120 elements (with 240 versors)
- all three are interesting groups, e.g. in **neutrino and flavour physics** for family symmetry model building

Spinors and Polytopes

- The space of $Cl(3)$ -spinors and quaternions have a **4D Euclidean signature**: $\psi = a_0 + a_i |e_i \Rightarrow \psi \tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- Can reinterpret **spinors in \mathbb{R}^3** as **vectors in \mathbb{R}^4**
- Then the spinors constitute the **vertices** of the **16-cell**, **24-cell**, **24-cell and dual 24-cell** and the **600-cell**
- These are 4D analogues of the **Platonic Solids**: regular convex 4-polytopes



Spinors, Polytopes and Root systems

- The 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell are in fact the root systems of $A_1 \times A_1 \times A_1 \times A_1$, D_4 , F_4 and H_4
- Exceptional phenomena: D_4 (triality, important in string theory), F_4 (largest lattice symmetry in 4D), H_4 (largest non-crystallographic symmetry)
- Exceptional D_4 and F_4 arise from series A_3 and B_3
- In fact, can strengthen this statement on inducing polytopes to statement on inducing root systems

Root systems in three and four dimensions

The **spinors** generated from the reflections contained in the respective **rank-3 Coxeter group** via the geometric product are realisations of the **binary polyhedral groups** Q , $2T$, $2O$ and $2I$, which were known to generate (mostly exceptional) **rank-4 groups**, but **not known why**, and why the ‘**mysterious symmetries**’.

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$	
A_3		$2T$	D_4	
B_3		$2O$	F_4	
H_3		$2I$	H_4	

Arnold's Trinities

Arnold's observation that many areas of real mathematics can be **complexified** and **quaternionified** resulting in theories with a similar structure.

- The **fundamental trinity** is thus $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The **projective spaces** $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The **spheres** $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The **Möbius/Hopf bundles** $(S^1 \rightarrow S^1, S^4 \rightarrow S^2, S^7 \rightarrow S^4)$
- The **Lie Algebras** (E_6, E_7, E_8)
- The symmetries of the **Platonic Solids** (A_3, B_3, H_3)
- The **4D groups** (D_4, F_4, H_4)
- **New connections** via my **Clifford spinor construction** (see McKay correspondence)

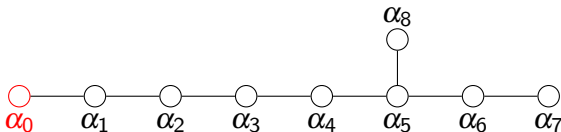
Platonic Trinities

- Arnold's connection between (A_3, B_3, H_3) and (D_4, F_4, H_4) is **very convoluted** and involves numerous other trinities at intermediate steps:
- **Decomposition of the projective plane** into Weyl chambers and Springer cones
- The **number of Weyl chambers** in each segment is $24 = 2(1 + 3 + 3 + 5)$, $48 = 2(1 + 5 + 7 + 11)$, $120 = 2(1 + 11 + 19 + 29)$
- Notice this miraculously **matches the quasihomogeneous weights** $((2, 4, 4, 6), (2, 6, 8, 12), (2, 12, 20, 30))$ of the Coxeter groups (D_4, F_4, H_4)
- Believe the Clifford connection is **more direct**

Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate **conjugacy classes** etc of versors in Clifford
- Chiral (**binary**) polyhedral groups have irreps
- tetrahedral (12/24): $1, 1', 1'', 2_s, 2'_s, 2''_s, 3$
- octahedral (24/48): $1, 1', 2, 2_s, 2'_s, 3, 3', 4_s$
- icosahedral (60/120): $1, 2_s, 2'_s, 3, \bar{3}, 4, 4_s, 5, 6_s$
- All binary are **discrete subgroups of $SU(2)$** and all thus have a 2_s spinor irrep
- Connection with the **McKay correspondence!**

Affine extensions – E_8^-



$$-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$$

AKA E_8^+ and along with E_8^{++} and E_8^{+++} thought to be the underlying symmetry of **String and M-theory**

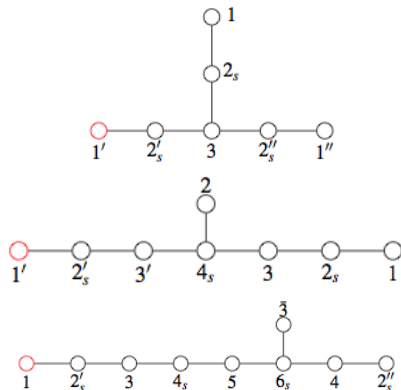
Also interesting from a pure mathematics point of view: **E_8 lattice**, **McKay correspondence** and **Monstrous Moonshine**.

The McKay Correspondence

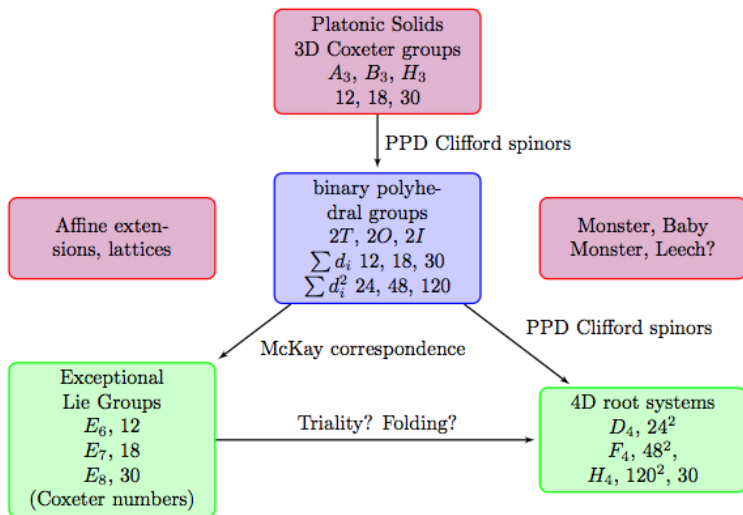
binary polyhe-
dral groups
 $2T, 2O, 2I$
 $\sum d_i$ 12, 18, 30
 $\sum d_i^2$ 24, 48, 120

McKay correspondence

Exceptional
Lie Groups
 E_6 , 12
 E_7 , 18
 E_8 , 30
(Coxeter numbers)



The McKay Correspondence



The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the **cyclic** and **dicyclic groups** are in correspondence with A_n and D_n , e.g. the quaternion group Q and D_4^+ . So McKay correspondence not just a trinity but **ADE-classification**. We also have $I_2(n)$ on top of the trinity (A_3, B_3, H_3)

rank-3 group	diagram	binary	rank-4 group	diagram	Lie algebra	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$		D_4^+	
A_3		$2T$	D_4		E_6^+	
B_3		$2O$	F_4		E_7^+	
H_3		$2I$	H_4		E_8^+	

4D geometry is surprisingly important for HEP

- 4D root systems are **surprisingly relevant to HEP**
- A_4 is $SU(5)$ and comes up in **Grand Unification**
- D_4 is $SO(8)$ and is the little group of **String theory**
- In particular, its **triality symmetry** is crucial for showing the equivalence of RNS and GS strings
- B_4 is $SO(9)$ and is the little group of **M-Theory**
- F_4 is the **largest crystallographic** symmetry in 4D and H_4 is the **largest non-crystallographic** group
- The above are **subgroups** of the latter two
- **Spinorial nature** of the root systems could have **surprising consequences for HEP**

Quaternions and Clifford Algebra

- The unit **spinors** $\{1; i; j; k\}$ of $\text{Cl}(3)$ are isomorphic to the **quaternion** algebra \mathbb{H} (up to sign)
- The 3D **Hodge dual of a vector** is a **pure bivector** which corresponds to a **pure quaternion**, and their products are identical (up to sign)

Discrete Quaternion groups

- The 8 quaternions of the form $(\pm 1, 0, 0, 0)$ and permutations are called the **Lipschitz units**, and form a realisation of the **quaternion group** in 8 elements.
- The 8 Lipschitz units together with $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ are called the **Hurwitz units**, and realise the **binary tetrahedral group** of order 24. Together with the 24 'dual' quaternions of the form $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$, they form a group isomorphic to the **binary octahedral group** of order 48.
- The 24 Hurwitz units together with the 96 unit quaternions of the form $(0, \pm \tau, \pm 1, \pm \sigma)$ and even permutations, are called the **Icosians**. The icosian group is isomorphic to the **binary icosahedral group** with 120 elements.

Quaternionic representations of 3D and 4D Coxeter groups

- Groups E_8 , D_4 , F_4 and H_4 have representations in terms of **quaternions**
- **Extensively used** in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why
- e.g. H_4 consists of 120 elements of the form $(\pm 1, 0, 0, 0)$, $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and $(0, \pm \tau, \pm 1, \pm \sigma)$
- Seen as remarkable that the **subset of the 30 pure quaternions** is a realisation of H_3 (**a sub-root system**)
- Similarly, A_3 , B_3 , $A_1 \times A_1 \times A_1$ have representations in terms of **pure quaternions**
- Will see there is a **much simpler geometric explanation**

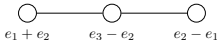
Quaternionic representations used in the literature



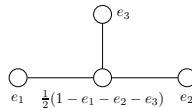
$A_1 \times A_1 \times A_1$



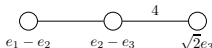
$A_1 \times A_1 \times A_1 \times A_1$



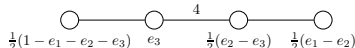
$A_3 = D_3$



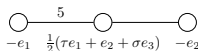
D_4



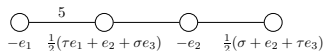
B_3



F_4



H_3



H_4

Demystifying Quaternionic Representations

- 3D: **Pure quaternions** = Hodge dualised (pseudoscalar) **root vectors**
- In fact, they are the **simple roots of the Coxeter groups**
- 4D: **Quaternions** = disguised **spinors** – but those of the **3D Coxeter group** i.e. the binary polyhedral groups!
- This relation between 3D and 4D via the geometric product does not seem to be known
- Quaternion multiplication = ordinary Clifford reflections and rotations

Demystifying Quaternionic Representations

- Pure quaternion subset of 4D groups only gives 3D group if the 3D group contains the inversion/pseudoscalar /
- e.g. does not work for the tetrahedral group A_3 , but $A_3 \rightarrow D_4$ induction still works, with the central node essentially 'spinorial'
- In fact, it goes the other way around: the 3D groups induce the 4D groups via spinors
- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as $R_1 = \alpha_1 \alpha_2$ and $R_2 = \alpha_2 \alpha_3$
- Can see these are 'spinor generators' and how they don't really contain any more information/roots than the rank-3 groups alone

Quaternions vs Clifford versors

- **Sandwiching** is often seen as particularly nice feature of the **quaternions giving rotations**
- This is actually a **general feature** of Clifford algebras/versors **in any dimension**; the isomorphism to the **quaternions** is **accidental** to 3D
- However, the **root system** construction does not necessarily generalise
- 2D generalisation merely gives that $I_2(n)$ is **self-dual**
- **Octonionic** generalisation just induces two copies of the above 4D root systems, e.g. $A_3 \rightarrow D_4 \oplus D_4$

References (single-author)

- Clifford algebra unveils a surprising geometric significance of quaternionic root systems of Coxeter groups
Advances in Applied Clifford Algebras, June 2013, Volume 23, Issue 2, pp 301-321
- A Clifford algebraic framework for Coxeter group theoretic computations (Conference Prize at AGACSE 2012)
Advances in Applied Clifford Algebras (2013)
- Nomination for W.K. Clifford Prize (2014)
- Invitation to Arizona State University
- Rank-3 root systems induce root systems of rank 4 via a new Clifford spinor construction arXiv:1207.7339 (2012)
- Platonic Solids generate their 4-dimensional analogues
Acta Cryst. A69 (2013)

Coxeter Elements, Degrees and Exponents

- Like the symmetric group, Coxeter groups can have **invariant polynomials**. Their **degrees** d are important invariants/group characteristics.
- Turns out that actually **degrees** d are intimately related to so-called **exponents** m $m = d - 1$.

Coxeter Elements, Degrees and Exponents

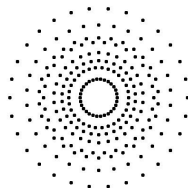
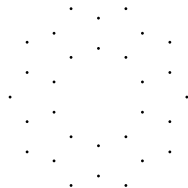
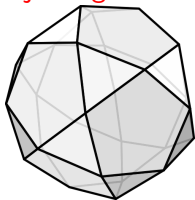
- A **Coxeter Element** is any combination of all the simple reflections $w = s_1 \dots s_n$, i.e. in Clifford algebra it is encoded by the versor $W = \alpha_1 \dots \alpha_n$ acting as $v \rightarrow wv = \pm \tilde{W} v W$. All such elements are conjugate and thus their **order** is invariant and called the **Coxeter number h** .
- The Coxeter element has **complex eigenvalues** of the form $\exp(2\pi mi/h)$ where m are called **exponents**.
- Standard theory **complexifies** the real Coxeter group situation in order to find **complex eigenvalues**, then takes **real sections** again (the unfortunate standard procedure in many situations) – without any insight into the complex structure (or in fact, there are different ones).

Coxeter Elements, Degrees and Exponents

- The Coxeter element has **complex eigenvalues** of the form $\exp(2\pi mi/h)$ where m are called **exponents**
- Standard theory **complexifies** the real Coxeter group situation in order to find **complex eigenvalues**, then takes **real** sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure(s)
- In particular, **1** and **$h-1$** are always exponents
- Turns out that actually **exponents and degrees** are intimately related ($m = d - 1$). The construction is slightly roundabout but uniform, and uses the **Coxeter plane**.

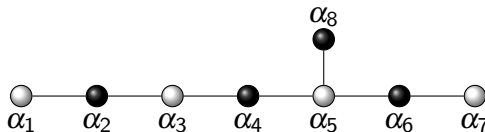
The Coxeter Plane

- Can show **every** (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by **projecting** their root system onto the Coxeter plane



The Coxeter Plane

- Obvious from Clifford point of view, that Coxeter element has eigenspaces (**eigenblades**) rather than just eigenvectors
- In particular, can show **every** (for our purposes) Coxeter group has a Coxeter plane
- Existence relies on the fact that all groups in question have **tree-like Dynkin diagrams**, and thus admit an **alternate colouring**
- Essentially just gives **two sets of mutually commuting generators**



The Coxeter Plane

- Existence relies on the fact that all groups in question have **tree-like Dynkin diagrams**, and thus admit an alternate colouring
- Essentially just gives **two sets of orthogonal = mutually commuting generators but anticommuting root vectors** α_w and α_b (duals ω)
- Cartan matrices are positive definite, and thus have a **Perron-Frobenius** (all positive) eigenvector λ_i .
- Take **linear combinations** of components of this eigenvector as coefficients of two vectors from the orthogonal sets
$$v_w = \sum \lambda_w \omega_w \text{ and } v_b = \sum \lambda_b \omega_b$$
- Their **outer product/Coxeter plane bivector** $B_C = v_b \wedge v_w$ describes an **invariant plane** where w acts by rotation by $2\pi/h$.

Clifford Algebra and the Coxeter Plane – 2D case

- For $I_2(n)$ take $\alpha_1 = e_1, \alpha_2 = -\cos \frac{\pi}{n} e_1 + \sin \frac{\pi}{n} e_2$

- So Coxeter versor is just

$$W = \alpha_1 \alpha_2 = -\cos \frac{\pi}{n} + \sin \frac{\pi}{n} e_1 e_2 = -\cos \frac{\pi}{n} + \sin \frac{\pi}{n} I = -\exp \left(-\frac{\pi I}{n} \right)$$

- In Clifford algebra it is therefore immediately obvious that the action of the $I_2(n)$ Coxeter element is described by a versor (here a rotor/spinor) that encodes rotations in the $e_1 e_2$ -Coxeter-plane and yields $h = n$ since trivially $W^n = (-1)^{n+1}$ yielding $w^n = 1$ via $wv = \tilde{W}vW$.

Clifford Algebra and the Coxeter Plane – 2D case

- So **Coxeter versor** is just $W = -\exp\left(-\frac{\pi I}{n}\right)$
- $I = e_1 e_2$ **anticommutes** with both e_1 and e_2 such that **sandwiching formula** becomes

$$v \rightarrow wv = \tilde{W}vW = \tilde{W}^2v = \exp\left(\pm \frac{2\pi I}{n}\right)v \text{ immediately}$$

yielding the standard result for the **complex eigenvalues** in real Clifford algebra **without any need for artificial complexification**

- The Coxeter plane bivector $B_C = e_1 e_2 = I$ gives the **complex structure**
- The Coxeter plane bivector B_C is invariant under the **Coxeter versor** $\tilde{W}B_CW = \pm B_C$.

Clifford Algebra and the Coxeter Plane – 3D case

- In 3D, A_3 , B_3 , H_3 have $\{1, 2, 3\}$, $\{1, 3, 5\}$ and $\{1, 5, 9\}$
- Coxeter element is product of a **spinor** in the Coxeter plane with the same complex structure as before, and a **reflection perpendicular** to the plane
- So in 3D still completely determined by the plane
- 1 and $h-1$ are **rotations** in **Coxeter plane**
- $h/2$ is the **reflection** (for v in the normal direction)

$$wv = \tilde{W}^2 = \exp\left(\pm \frac{2\pi I}{h} \frac{h}{2}\right) = \exp(\pm \pi I)v = -v$$

Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can **factorise** W into **orthogonal** (commuting/anticommuting) components

$$W = \alpha_1 \dots \alpha_n = W_1 \dots W_n \text{ with } W_i = \exp(\pi m_i l_i / h)$$
- Here, l_i is a bivector describing a **plane** with $l_i^2 = -1$
- For v **orthogonal to the plane** described by l_i we have

$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i W_i v = v \text{ so cancels out}$$
- For v **in the plane** we have

$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i l_i / h) v$$
- Thus if we **decompose** W into **orthogonal eigenspaces**, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

Clifford algebra: no need for complexification

- For v in the plane we have

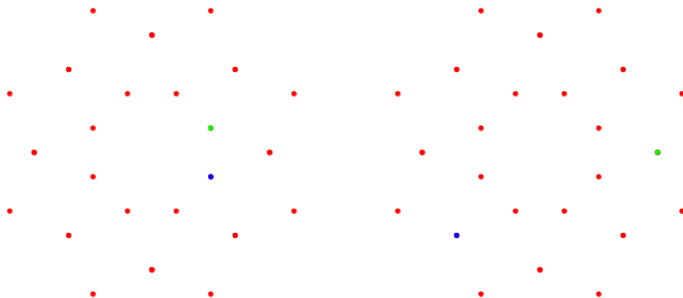
$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i l_i / h) v$$

- So **complex eigenvalue equation** arises geometrically **without any need** for complexification
- **Different complex structures** immediately give different **eigenplanes**
- Eigenvalues/angles/**exponents** given from just factorising $W = \alpha_1 \dots \alpha_n$
- E.g. B_4 has exponents 1, 3, 5, 7 and $W = \exp\left(\frac{\pi}{8} l_1\right) \exp\left(\frac{3\pi}{8} l_2\right)$
- Here we have been looking for orthogonal eigenspaces, so **innocuous** – different complex structures commute
- But not in general – **naive complexification** can be misleading

Clifford Algebra and the Coxeter Plane – 4D case

- E.g. B_4 has exponents 1, 3, 5, 7
- Coxeter versor decomposes into **orthogonal components**

$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} I B_C\right)$$



Clifford Algebra and the Coxeter Plane – 4D case

rank 4	exponents	W-factorisation
A_4	1, 2, 3, 4	$W = \exp\left(\frac{\pi}{5} B_C\right) \exp\left(\frac{2\pi}{5} I B_C\right)$
B_4	1, 3, 5, 7	$W = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} I B_C\right)$
D_4	1, 3, 3, 5	$W = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{\pi}{2} I B_C\right)$
F_4	1, 5, 7, 11	$W = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} I B_C\right)$
H_4	1, 11, 19, 29	$W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$

Actually, in 2, 3 and 4 dimensions it couldn't really be any other way

Clifford Algebra and the Coxeter Plane – D_6

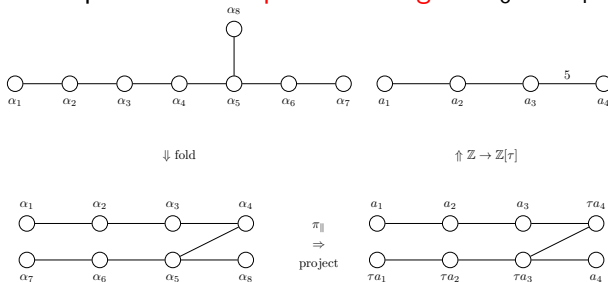
- For D_6 one has exponents $\boxed{1, 3, 5, 5, 7, 9}$
- Coxeter versor decomposes into orthogonal bits as

$$W = \frac{1}{\sqrt{5}}(e_1 + e_2 + e_3 - e_4 - e_5)e_6 \exp\left(\frac{\pi}{10}B_C\right) \exp\left(\frac{3\pi}{10}B_2\right)$$

- Now **bivector exponentials** correspond to **rotations in orthogonal planes**
- **Vector** factors correspond to **reflections**
- For odd n , there is always **one such vector factor** in D_n , and for even n there are **two**

Projection and Diagram Foldings

Compare with the 'partial folding' of E_8 to H_4



$$s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}, s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}, s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}, s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4$$

Imaginary differences – different imaginaries

So what has been **gained** by this **Clifford view**?

- There are **different** entities that serve as **unit imaginaries**
- They have a **geometric** interpretation as an **eigenplane of the Coxeter element**
- These don't need to **commute** with everything like i (though they do here – at least anticommute. But that is because we looked for **orthogonal decompositions**)
- But see that in general **naive complexification** can be a dangerous thing to do – **unnecessary**, issues of **commutativity**, **confusing** different imaginaries etc

Conformal geometry and Clifford algebra

- The conformal group $C(p, q) \sim SO(p+1, q+1)$
- So can use **versor representation** of conformal transformations in Clifford algebra (**reflections, translations, inversions ...**)
- Treat all of them **multiplicatively** in terms of versors and use **sandwiching** $Ax\tilde{A}$
- E.g. can generate a whole **root lattice** multiplicatively with **compact reflection part** and **translations**

Conformal Clifford Algebra

- The **conformal group** $C(n, p)$ is homomorphic to $Spin(n+1, p+1)$
- Go to $e_1, e_2, e_3, e, \bar{e}$, with $e_i^2 = 1, e^2 = 1, \bar{e}^2 = -1$
- Define two null vectors $n \equiv e + \bar{e}, \bar{n} \equiv e - \bar{e}$
- Can embed the 3D vector $x = x^\mu e_\mu = xe_1 + ye_2 + ze_3$ as a **null vector in 5D** ($\hat{X} \cdot n = -1$)

$$F(x) \equiv \hat{X} = \frac{1}{2\lambda^2}(x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

- Essentially **linear** action of $SO(n+1, p+1)$ in embedding space induces a **non-linear** realisation of the conformal group on the **projective light cone** (Dirac/Hestenes/Lasenby)
- So neat thing is that **conformal transformations** are now done by **rotors** (except inversion which is a reflection) – distances are given by **inner products**

Operations in Conformal Geometric Algebra

- **Amsterdam protocol**: $e = e_+$, $\bar{e} = e_-$, $n = n_\infty$ and $\bar{n} = n_0$.
- **Reflections** $y' = -xyx$ since e and $\bar{e} \Rightarrow n$ and \bar{n} are orthogonal to $x \Rightarrow$ anticommute $-xnx = n$ and $-x\bar{n}x = \bar{n}$:

$$-xF(y)x = F(y') = F(-xyx)$$

- **Rotations** $y' = Ry\tilde{R}$ from reflections via **Cartan-Dieudonné**

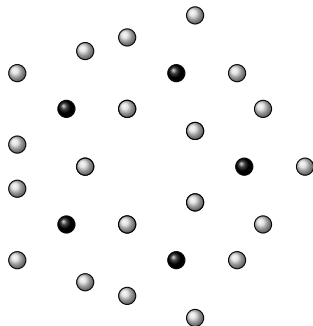
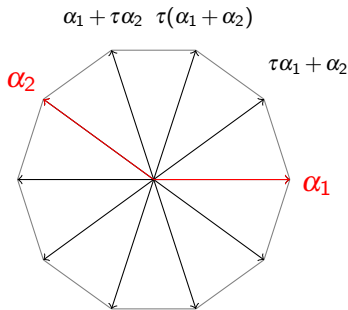
$$RF(y)\tilde{R} = F(y') = F(Ry\tilde{R})$$

- **Translations** $y' = y + a$ rotor $T_a = \exp\left(\frac{na}{2\lambda}\right) = 1 + \frac{na}{2\lambda}$

$$T_a F(y) \tilde{T}_a = F(y') = F(y + a)$$

Proof of Principle

Construction of **root systems** and **quasicrystalline point arrays** carries through, e.g. here for H_2 and a pentagon with translation $1/\tau$



Benefits of this approach

- **Conceptual Unification** of Rotations and Translations via rotors
- Construct **root system** from the simple roots as before, and likewise for **quasicrystalline point arrays**
- Increased **numerical stability** (not really an issue here) due to projective representation

A new set of Bianchi IX Killing Vectors

- Used **Conformal Clifford algebra** setup to treat conformal group $C(1,3)$ as $SO(2,4)$
- Stabiliser subgroup of a certain vector gives the **de Sitter** group (Killing vectors)
- Using a certain **projection** broke this down to **two commuting $SU(2) \times SU(2)$**
- This is a **new set of Bianchi IX Killing vectors** with **nice symmetry properties**

A unified framework for polyhedral groups

Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm \tilde{A}xA$
$Spin(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$Pin(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the **chiral icosahedral** group has 60 elements, encoded in Clifford by 120 rotors, which form the **binary icosahedral** group
- together with the **inversion/pseudoscalar** I this gives 60 rotations and 60 rotoinversions, i.e. the **full icosahedral** group H_3 in 120 elements (with 240 versors)
- all three are interesting groups, e.g. in **neutrino and flavour physics** for family symmetry model building

Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate **conjugacy classes** etc of versors in Clifford
- Chiral (**binary**) polyhedral groups have irreps
- tetrahedral (12/24): $1, 1', 1'', 2_s, 2'_s, 2''_s, 3$
- octahedral (24/48): $1, 1', 2, 2_s, 2'_s, 3, 3', 4_s$
- icosahedral (60/120): $1, 2_s, 2'_s, 3, \bar{3}, 4, 4_s, 5, 6_s$
- All binary are **discrete subgroups of $SU(2)$** and all thus have a 2_s spinor irrep
- See **McKay correspondence**
- Interesting to look at spinors/binary groups in their own right
– see **Induction Theorem**

Some Group Theory: chiral, full, binary, pin

- Full (**Coxeter**) is just **two copies** of this (24/48/120 i.e. same order as binary since both $\text{Spin}(3)$ and $O(3)$ are double covers of $SO(3)$)
- **Pin group** is just $1 + I$ of this for B_3 and H_3 , which contain the inversion I
- but **not for A_3 !** (which doesn't – c.f. quaternionic reps)
- Instead $\text{Pin}(A_3)$ has the same conjugacy classes as $\text{Spin}(B_3)$

Conjugacy Classes: Quaternion group Q

- Five conjugacy classes: $\{1\}$, $\{-1\}$, $\{\pm e_1 e_2\}$, $\{\pm e_2 e_3\}$, $\{\pm e_3 e_1\}$
- Different conjugacy classes correspond to different geometric subspaces in the Clifford algebra
- Bit trivial for the quaternion group, but extends to arbitrary dimension

Conjugacy Classes: Binary octahedral group $2O$

- Eight conjugacy classes: $\{1\}$, $\{-1\}$, $\underline{6}$: **bivectors** $\{\pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1\}$; $\underline{6}'$: **bivector exponentials** $\exp \underline{6}$; $\underline{6}''$: $\exp -\underline{6}$; $\underline{8}$: **spinors** $\{1 \pm e_1 e_2 \pm e_2 e_3 \pm e_3 e_1, \dots\}$; $\underline{8}'$: $-\underline{8}$; $\underline{12}$: **bivectors** $\{e_1(e_2 + e_3), \dots\}$;
- Turns out most of these are the **same as for $\text{Pin}(A_3)$** , and the remaining ones can be mapped to each other
- Though in $\text{Pin}(A_3)$ also have **odd grade** elements, so some of the conjugacy classes are vector+trivector etc, i.e. different geometric interpretation

Character tables: Quaternion group Q (from $A_1^3 \rightarrow A_1^4$)

Q	1	-1	$\pm i$	$\pm j$	$\pm k$
1	1	1	1	1	1
$1'$	1	1	1	-1	-1
$1''$	1	1	-1	1	-1
$1'''$	1	1	-1	-1	1
2	2	-2	0	0	0
4	4	-4	0	0	0

Latter is of **quaternionic type** – somehow seen as particularly noteworthy

Character tables: binary octahedral group $2O$ (from $B_3 \rightarrow F_4$)

$2O$	1	1	6	8	8	6	6	12
1	1	1	1	1	1	1	1	1
$1'$	1	1	1	1	1	-1	-1	-1
2	2	2	2	-1	-1	0	0	0
3	3	3	-1	0	0	1	1	-1
$3'$	3	3	-1	0	0	-1	-1	1
4	4	-4	0	2	-2	$2\sqrt{2}$	$-2\sqrt{2}$	0
$4'$	4	-4	0	2	-2	$-2\sqrt{2}$	$2\sqrt{2}$	0
8	8	-8	0	-2	2	0	0	0

Again some of quaternionic type

Representations

- This **Clifford multivector construction** of the polyhedral groups is a **faithful realisation/representation**, i.e. is essentially the same as the abstract group
- But can define several different **representations** from these versor groups (may or may not be **irreducible** ones)
- Representations: matrices $D(R)$ such that

$$D(R_1 R_2) = D(R_1) D(R_2)$$

Representations

- Representations: matrices $D(R)$ such that

$$D(R_1 R_2) = D(R_1) D(R_2)$$

- **Trivial** representation: $D(R) = R1\tilde{R} = 1$
- **Rotation** representation: for nD vector $x = \sum a_i e_i$:

$$D(R)\underline{x} = R x \tilde{R} \quad \text{usual } SO(n) \text{ } n \times n\text{-matrix}$$

- Full representation: for nD vector $x = \sum a_i e_i$:

$$D(A)\underline{x} = A x \tilde{A}$$

usual $O(n)$ $n \times n$ -matrix

- **Spinor** representation: for nD spinor y (2^{n-1} components):

$$D(R)\underline{y} = R y \quad \text{a } 2^{n-1} \times 2^{n-1}\text{-matrix}$$

- **Versor** representation: for nD versor z (2^n components):

$$D(A)\underline{z} = A z \quad \text{a } 2^n \times 2^n\text{-matrix}$$

Character tables and Clifford reps: quaternion group Q

The **spinor** representation $D(R)\underline{y} = R\underline{y}$ of the quaternion group Q gives the representation of **quaternionic type**. (The **trace** of $D(R)$ is the **character**.)

Again just seen to be a consequence of the **accidental isomorphism** between 3D spinors and quaternions.

Q	1	-1	$\pm i$	$\pm j$	$\pm k$
1	1	1	1	1	1
1'	1	1	1	-1	-1
1''	1	1	-1	1	-1
1'''	1	1	-1	-1	1
2	2	-2	0	0	0
4	4	-4	0	0	0

Character tables and Clifford reps: binary octahedral group $2O$

The **spinor** representation $D(R)\underline{y} = Ry$ of the quaternion group $2O$ gives the irrep of **quaternionic type**.

The **rotation** representation $D(R)\underline{x} = Rx\tilde{R}$ gives **3 irrep**.

$2O$	1	1	6	8	8	6	6	12
1	1	1	1	1	1	1	1	1
$1'$	1	1	1	1	1	-1	-1	-1
2	2	2	2	-1	-1	0	0	0
3	3	3	-1	0	0	1	1	-1
$3'$	3	3	-1	0	0	-1	-1	1
4	4	-4	0	2	-2	$2\sqrt{2}$	$-2\sqrt{2}$	0
$4'$	4	-4	0	2	-2	$-2\sqrt{2}$	$2\sqrt{2}$	0
8	8	-8	0	-2	2	0	0	0

Clifford: groups and representations summary

- Clifford algebra provides a **unified framework** for chiral/full/binary/pin **polyhedral** groups all in the **same** representation/realisation space/**algebra** (c.f. usual $SO(3)$ vs $SU(2)$ representations)
- Structure of the algebra \Rightarrow different **conjugacy classes** are **different kinds of objects in the algebra** and are kept separate
- Several **representations** follow, in particular have **geometric insight** into **complex and quaternionic** representations

Clifford summary

- Interesting **induction theorem** linking geometry of 3D and 4D
- Geometric **complex structures** and **non-trivial commutativity** properties
- Simple versor representation of **orthogonal transformations**
- **Conformal** geometry
- Some interesting new results on **group and representation theory**
- **Lie algebras** can be constructed in Clifford algebra as **bivector algebras**, and **Lie groups** as **spin groups** (work with Phoenix)

1 Introduction

- Coxeter groups and root systems
- Clifford algebras

2 Coxeter and Clifford

- The Induction Theorem – from 3D to 4D
- The Coxeter Plane
- Conformal Geometry
- Some Group Theory

3 Moonshine and Outlook

Monstrous Moonshine

- **Mysterious connection** between two very different areas of Mathematics
- **Modular forms** (functions that live on a torus with complex structure τ): Fourier expansion **coefficients** wrt ($q = e^{2\pi i\tau}$)
- **Finite simple groups**: **dimensions** of irreducible representations
- **Monstrous Moonshine**: The largest sporadic group, the **Monster M** ($\{1, 196883, 21296876, \dots\}$) and the Klein **$j(\tau)$** modular function
- $j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots$
- $196884 = 196883 + 1, 21493760 = 21296876 + 196883 + 1, \dots$

Mathieu Moonshine

- Similar Moonshine phenomenon
- Modular form: elliptic genus of an $\mathcal{N} = 4$ SCFT compactified on a K3-surface
- Finite simple group: Mathieu M_{24}
($\{45, 231, 770, 2277, 5796 \dots\}$)
- Elliptic genus is

$$E_{K3}(\tau, z) = -2Ch(0; \tau, z) + 20Ch(1/2; \tau, z) + e(q)Ch(\tau, z)$$

where all the coefficients in the q -series

$$e(q) = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + \dots$$

are twice the dimension of some M_{24} irrep

Clifford and Moonshine

- Looking at **Wess-Zumino-Witten models**, i.e. strings propagating on a Lie group manifold
- Condition of **extended supersymmetry** ultimately hinges on **classification** of Clifford algebras – deep connection
- Connections with **binary polyhedral groups, Monstrous Moonshine, McKay correspondence, lattices, affine extensions, Lie groups/algebras** etc
- **Elliptic genus** is constant on (connected components of) the **moduli space**: better understanding as a **topological** feature as the **index of a Dirac operator**?

Thank you!

Back to the roots

- Unifying principles of **geometry** and **symmetry**
- **Discrete** groups (finite simple groups, polyhedral groups) and **continuous** groups (string compactifications, non-linear σ -models)
- **Exceptional phenomena**: E_8 , $H_4/F_4/D_4$ (spinorial), McKay correspondence, Monster M , Mathieu M_{24} ...
- **Applications**: from **mundane** (viruses, fullerenes, quasicrystals) to **exotic** (HEP, Moonshine) – **same mathematical tools**: Coxeter, Clifford, affine extensions etc

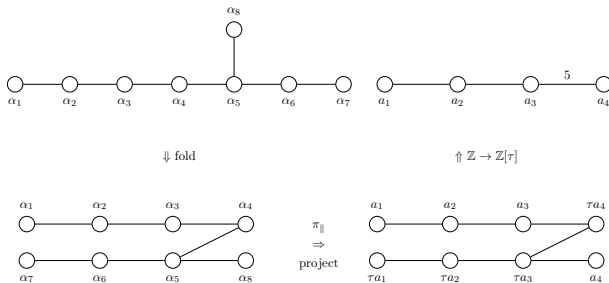
Mathematical Aspects: Better understanding of geometrical structures

- Affine extensions and translations – lattices, quasicrystals and projections
- Coxeter groups and Kac-Moody theory
- Euclidean, spherical, hyperbolic and conformal geometry
- Clifford and spinorial geometry
- Group theory, Lie groups and algebras (with Phoenix)
- Mathieu Moonshine, McKay correspondence
- Implications for the real world

HEP

- Gravitational and cosmological singularities – hidden symmetries
- New uses for non-crystallographic groups
- Topological defects
- Integrable systems
- Family symmetries (flavour/neutrino physics)
- 4D groups: A_4 , B_4 , $D_4 - F_4$, H_4 ?
- Spinor geometry
- E_8 features prominently – relation with H_4

Projection and Diagram Foldings



$$s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}, \quad s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}, \quad s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}, \quad s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4$$

E_8 has two H_4 -invariant subspaces – blockdiagonal form

D_6 has two H_3 -invariant subspaces

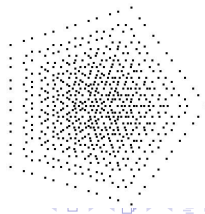
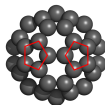
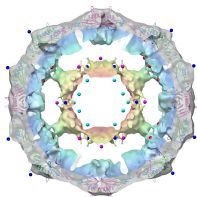
A_4 has two H_2 -invariant subspaces

4D geometry is surprisingly important for HEP

- 4D root systems are **surprisingly relevant to HEP**
- A_4 is $SU(5)$ and comes up in **Grand Unification**
- D_4 is $SO(8)$ and is the little group of **String theory**
- In particular, its **triality symmetry** is crucial for showing the equivalence of RNS and GS strings
- B_4 is $SO(9)$ and is the little group of **M-Theory**
- F_4 is the **largest crystallographic** symmetry in 4D and H_4 is the **largest non-crystallographic** group
- The above are **subgroups** of the latter two
- **Spinorial nature** of the root systems could have **surprising consequences for HEP**

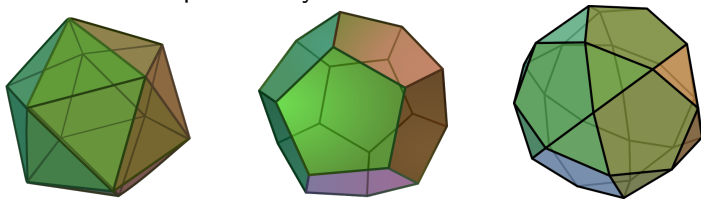
Motivation: Viruses

- Geometry of **polyhedra** described by **Coxeter** groups
- Viruses have to be '**economical**' with their **genes**
- Encode **structure** modulo **symmetry**
- **Largest discrete symmetry of space** is the **icosahedral** group
- Many other '**maximally symmetric**' objects in nature are also icosahedral: **Fullerenes & Quasicrystals**
- But: viruses are not just polyhedral – they have **radial structure**. **Affine extensions** give **translations**



Extend icosahedral group with distinguished translations

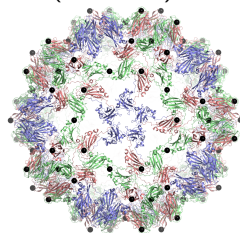
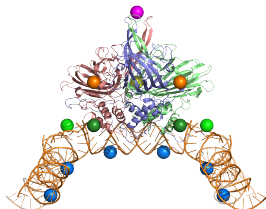
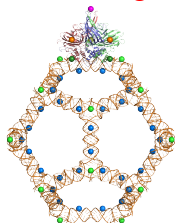
- Radial layers are **simultaneously constrained** by affine symmetry
- Works very well in practice: **finite library of blueprints**
- **Select** blueprint from the **outer shape** (capsid)
- Can **predict inner structure** (nucleic acid distribution) of the virus from the point array



Affine extensions of the icosahedral group (giving translations) and their **classification**.

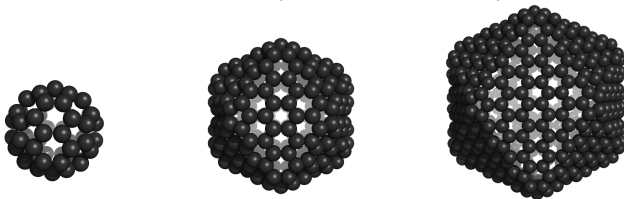
Use in Mathematical Virology

- Suffice to say **point arrays work very exceedingly well** in practice. Two papers on the mathematical (Coxeter) aspects.
- **Implemented computational problem in Clifford** – some **very interesting mathematics** comes out as well (see later).



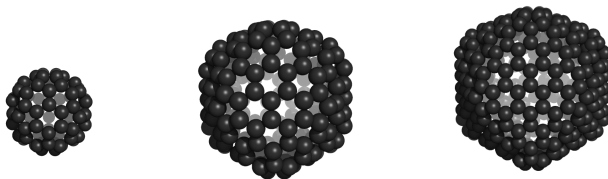
Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach: **carbon onions** ($C_{60} - C_{240} - C_{540}$)



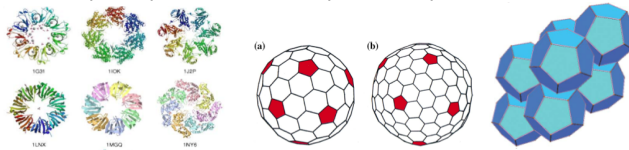
Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach: **carbon onions** ($C_{80} - C_{180} - C_{320}$)



Applied areas (EPSRC proposal)

- **Viruses:** Extend to large viruses; interesting results on **higher-order translations**
- **Proteins:** Extend affine symmetry to **$2D$** and apply to (**chiral**) proteins
- **Fullerenes:** Extend to larger fullerenes, in particular **chiral carbon onions**
- **Packings:** Novel analytical and numerical approaches to packings of polyhedral solids; with Colapinto (Santa Barbara), Twarock (York) and Thorpe (Phoenix)



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