A 3D spinorial view of the exceptional root systems

Pierre-Philippe Dechant

Department of Mathematics, University of York

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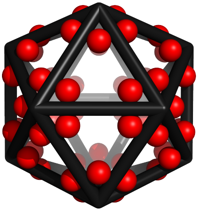
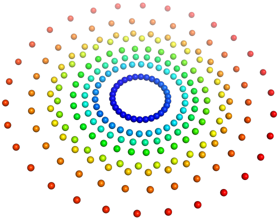
Main results

- Each 3D root system induces a 4D root system
- H_3 (icosahedral symmetry) induces the E_8 root system
- Clifford algebra is a very natural framework for root systems and reflection groups





Overview

1. Root systems and Clifford algebras
   - Root systems
   - Clifford Basics
2. H_4 as a rotation group I: 3D to 4D spinor induction, Trinities and McKay correspondence
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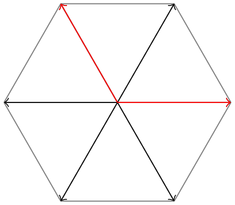
   - Trinities and McKay correspondence
3. E_8 from the icosahedron
4. H_4 as a rotation group II: The Coxeter plane

Root systems



Root system Φ: set of vectors α in a vector space with an inner product such that

1. $\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall\, \alpha \in \Phi}$
2. $\boxed{s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi}$

Simple roots: express every element of Φ via a $\mathbb{Z}$-linear combination.

reflection/Coxeter groups $\boxed{s_\alpha : v \to s_\alpha(v) = v - 2\frac{(v|\alpha)}{(\alpha|\alpha)}\alpha}$

Cartan Matrices

Cartan matrix of α_is is $\boxed{A_{ij} = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} = 2\frac{|\alpha_j|}{|\alpha_i|}\cos\theta_{ij}}$

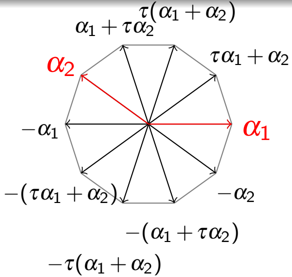
$$A_2\text{:}\ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

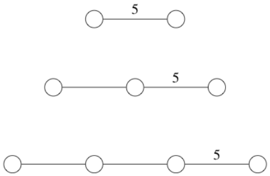
Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal, simple link = roots at $\frac{\pi}{3}$, link with label m = angle $\frac{\pi}{m}$.

Lie groups to Lie algebras to Coxeter groups to root systems

- Lie group: manifold of continuous symmetries (gauge theories, spacetime)
- Lie algebra: infinitesimal version near the identity
- Non-trivial part is given by a root lattice
- Weyl group is a crystallographic Coxeter group: $\boxed{A_n, B_n/C_n, D_n, G_2, F_4, E_6, E_7, E_8}$ generated by a root system.
- So via this route root systems are always crystallographic. Neglect non-crystallographic root systems $\boxed{I_2(n), H_3, H_4}$.

Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$





$$A=\begin{pmatrix}2 & -\tau\\ -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -\tau\\ 0 & -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -\tau\\ 0 & 0 & -\tau & 2\end{pmatrix}$$

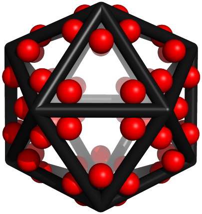
$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate rotational symmetries of order 5

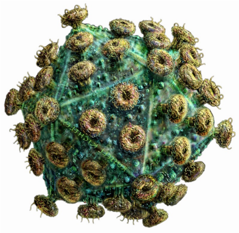
linear combinations now in the extended integer ring

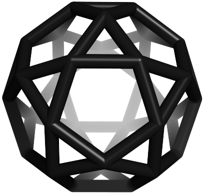
$\mathbb{Z}[\tau]=\{a+\tau b | a,b\in\mathbb{Z}\}$ golden ratio $\tau=\frac{1}{2}(1+\sqrt{5})=2\cos\frac{\pi}{5}$

$x^2=x+1$ $\tau'=\sigma=\frac{1}{2}(1-\sqrt{5})=2\cos\frac{2\pi}{5}$ $\tau+\sigma=1, \tau\sigma=-1$

The Icosahedron







- Rotational icosahedral group is $I = A_5$ of order 60
- Full icosahedral group is H_3 of order 120 (including reflections/inversion); generated by the root system icosidodecahedron

Clifford Algebra and orthogonal transformations

- Form an algebra using the Geometric Product for two vectors

$$\boxed{ab \equiv a \cdot b + a \wedge b}$$

- Inner product is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting a in b is given by $\boxed{a' = a - 2(a \cdot b)b = -bab}$ (b and $-b$ doubly cover the same reflection)
- Via Cartan-Dieudonné theorem any orthogonal (/conformal/modular) transformation can be written as successive reflections

$$\boxed{x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1} = \pm A x \tilde{A}$$

Clifford Algebra of 3D

- E.g. Pauli algebra in 3D (likewise for Dirac algebra in 4D) is

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

- We can multiply together root vectors in this algebra $\alpha_i\alpha_j\ldots$
- A general element has 8 components, even products (rotations/spinors) have four components:

$$\boxed{R = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2 \Rightarrow R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

- So behaves as a 4D Euclidean object – inner product

$$\boxed{(R_1, R_2) = \frac{1}{2}(R_2\tilde{R}_1 + R_1\tilde{R}_2)}$$

Overview

Induction Theorem – root systems

- Theorem: 3D spinor groups give 4D root systems.

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- Check axioms:
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Induction Theorem – root systems

- Theorem: 3D spinor groups give 4D root systems.
- Check axioms:
  1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall \alpha \in \Phi$
  2. $s_\alpha \Phi = \Phi \ \forall \alpha \in \Phi$
- Proof: 1. R and $-R$ are in a spinor group by construction (double cover of orthogonal transformations), 2. closure under reflections is guaranteed by the closure property of the spinor group (with a twist: $-R_1\tilde{R}_2R_1$)
- Induction Theorem: Every rank-3 root system induces a rank-4 root system (and thereby Coxeter groups)

Spinors from reflections

- The 6 roots in $A_1 \times A_1 \times A_1$ generate 8 spinors.
- $\pm e_1$, $\pm e_2$, $\pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The discrete spinor group is isomorphic to the quaternion group Q.
- As 4D vectors these are the 8 roots of $A_1 \times A_1 \times A_1 \times A_1$ (the 16-cell).

H_4 as a rotation group I: as icosahedral spinors

- The H_3 root system has 30 roots e.g. simple roots $\alpha_1 = e_2, \alpha_2 = -\frac{1}{2}((\tau-1)e_1 + e_2 + \tau e_3)$ and $\alpha_3 = e_3$.
- The subgroup of rotations is A_5 of order 60
- These are doubly covered by 120 spinors of the form $\alpha_1\alpha_2 = -\frac{1}{2}(1-(\tau-1)e_1e_2 + \tau e_2e_3)$, $\alpha_1\alpha_3 = e_2e_3$ and $\alpha_2\alpha_3 = -\frac{1}{2}(\tau - (\tau-1)e_3e_1 + e_2e_3)$.
- As a set of vectors in 4D, they are

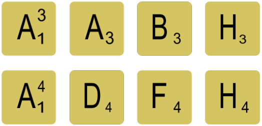
$$(\pm 1, 0, 0, 0) \text{ (8 permutations) }, \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) \text{ (16 permutations)}$$

$$\frac{1}{2}(0, \pm 1, \pm\sigma, \pm\tau) \text{ (96 even permutations) },$$

which are precisely the 120 roots of the H_4 root system.

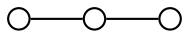
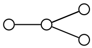
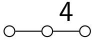
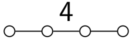
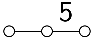
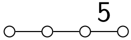
Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the Platonic Solids:
- The 6/12/18/30 roots in $A_1 \times A_1 \times A_1/A_3/B_3/H_3$ generate 8/24/48/120 spinors.
- E.g. $\pm e_1$, $\pm e_2$, $\pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The discrete spinor group is isomorphic to the quaternion group Q / binary tetrahedral group $2T$/ binary octahedral group $2O$/ binary icosahedral group $2I$).



Exceptional Root Systems

- Exceptional phenomena: D_4 (triality, important in string theory), F_4 (largest lattice symmetry in 4D), H_4 (largest non-crystallographic symmetry); Exceptional D_4 and F_4 arise from series A_3 and B_3

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$	
A_3		$2T$	D_4	
B_3	 4	$2O$	F_4	 4
H_3	 5	$2I$	H_4	 5

Arnold's Trinities

Arnold's observation that many areas of real mathematics can be complexified and quaternionified resulting in theories with a similar structure.

- The fundamental trinity is thus $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The projective spaces $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The spheres $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The Möbius/Hopf bundles $(S^1 \to S^1, S^4 \to S^2, S^7 \to S^4)$
- The Lie Algebras (E_6, E_7, E_8)
- The symmetries of the Platonic Solids (A_3, B_3, H_3)
- The 4D groups (D_4, F_4, H_4)
- New connections via my Clifford spinor construction (see McKay correspondence)

Platonic Trinities

- Arnold's connection between (A_3, B_3, H_3) and (D_4, F_4, H_4) is very convoluted and involves numerous other trinities at intermediate steps:
- Decomposition of the projective plane into Weyl chambers and Springer cones
- The number of Weyl chambers in each segment is $24 = 2(1+3+3+5), 48 = 2(1+5+7+11), 120 = 2(1+11+19+29)$
- Notice this miraculously matches the quasihomogeneous weights $((2,4,4,6),(2,6,8,12),(2,12,20,30))$ of the Coxeter groups (D_4, F_4, H_4)
- Believe the Clifford connection is more direct

A unified framework for polyhedral groups

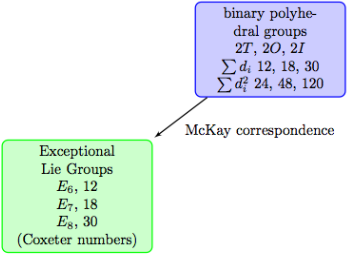
Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm\tilde{A}xA$
$\mathrm{Spin}(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$\mathrm{Pin}(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

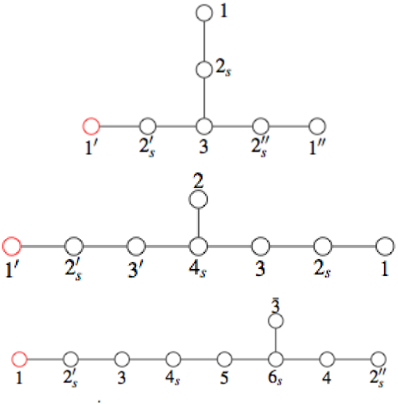
- e.g. the chiral icosahedral group has 60 elements, encoded in Clifford by 120 spinors, which form the binary icosahedral group
- together with the inversion/pseudoscalar I this gives 60 rotations and 60 rotoinversions, i.e. the full icosahedral group H_3 in 120 elements (with 240 pinors)

Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate conjugacy classes etc of pinors in Clifford algebra
- Chiral (binary) polyhedral groups have irreps
- tetrahedral (12/24): 1, $1'$, $1''$, 2_s, $2'_s$, $2''_s$, 3
- octahedral (24/48): 1, $1'$, 2, 2_s, $2'_s$, 3, $3'$, 4_s
- icosahedral (60/120): 1, 2_s, $2'_s$, 3, $\bar{3}$, 4, 4_s, 5, 6_s
- Binary groups are discrete subgroups of $SU(2)$ and all thus have a 2_s spinor irrep
- Connection with the McKay correspondence!

The McKay Correspondence: Coxeter number, dimensions of irreps and tensor product graphs





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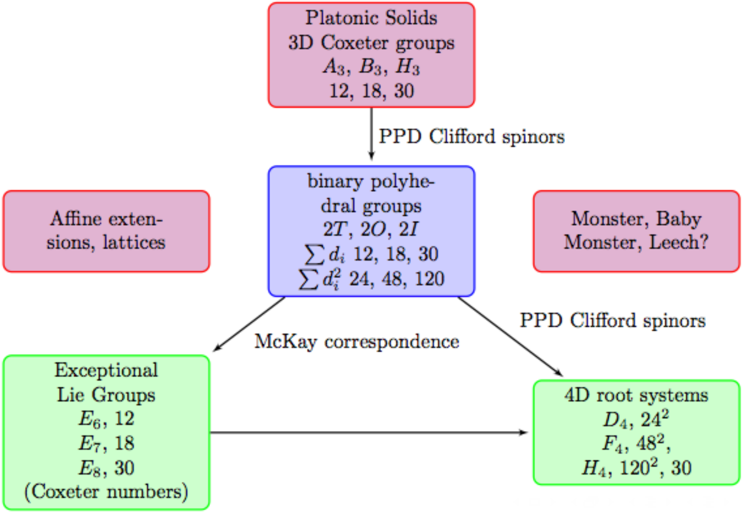
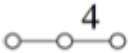
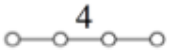
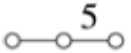
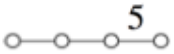
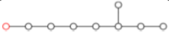


The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the cyclic and dicyclic groups are in correspondence with A_n and D_n, e.g. the quaternion group Q and D_4^+. So McKay correspondence not just a trinity but ADE-classification. We also have $I_2(n)$ on top of the trinity (A_3, B_3, H_3)

rank-3 group	diagram	binary	rank-4 group	diagram	Lie algebra	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$		D_4^+	
A_3		$2T$	D_4		E_6^+	
B_3		$2O$	F_4		E_7^+	
H_3		$2I$	H_4		E_8^+	

An indirect connection between E_8 and H_3?

- Trinities:

$$(12, 18, 30)$$

$$(A_3, B_3, H_3)$$

$$(2T, 2O, 2I)$$

$$(D_4, F_4, H_4)$$

$$(E_6, E_7, E_8)$$

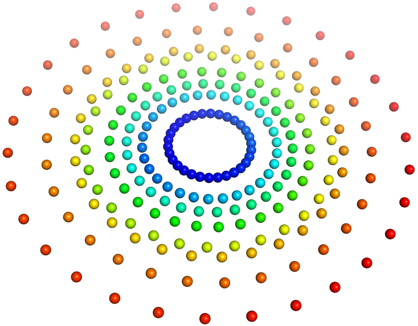
4D geometry is surprisingly important for HEP

- 4D root systems are surprisingly relevant to HEP
- A_4 is $SU(5)$ and comes up in Grand Unification
- D_4 is $SO(8)$ and is the little group of String theory
- In particular, its triality symmetry is crucial for showing the equivalence of RNS and GS strings
- B_4 is $SO(9)$ and is the little group of M-Theory
- F_4 is the largest crystallographic symmetry in 4D and H_4 is the largest non-crystallographic group
- The above are subgroups of the latter two
- Spinorial nature of the root systems could have surprising consequences for HEP

Overview

Exceptional E_8 (projected into the Coxeter plane)

- E_8 root system has 240 roots, H_3 has order 120



Exceptional E_8 – from the icosahedron

- Saw even products of the 30 roots of H_3 gave 120 spinors which in turn gave H_4 root system
- Taking all products gives group of 240 pinors with 8 components
- Essentially the inversion I just doubles the spinors

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

$$\boxed{R = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2 \& IR = b_0e_1e_2e_3 + b_1e_1 + b_2e_2 + b_3e_3}$$

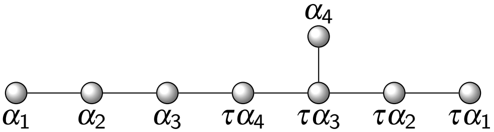
- Most intuitive inner product on the pinors gives only $H_4 \oplus H_4$
- But slightly more technical inner product gives precisely the E_8 root system from the icosahedron!

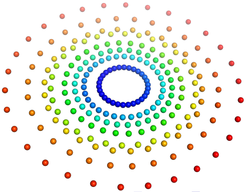
- Order 120 group H_3 doubly covered by 240 (s)pinors

- Essentially $H_4 + IH_4$, two sets of 120

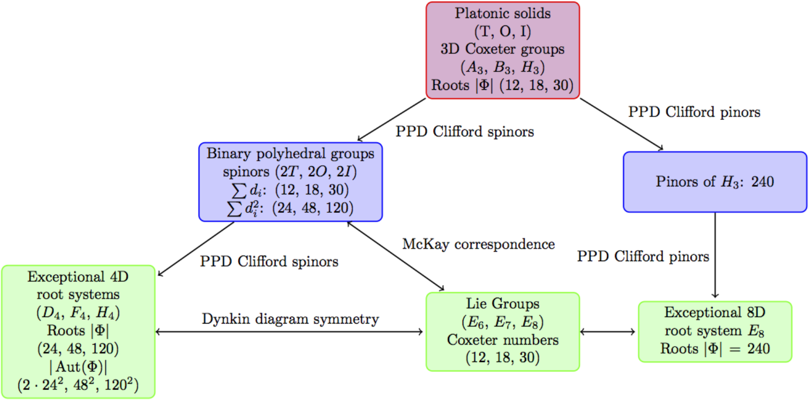
- Multiply second set by τI, take inner products, taking into account $\tau^2 = \tau + 1$, but THEN: set $\tau \to 0$! Each inner product is $(\alpha_i, \alpha_j) = a + \tau b \to (\alpha_i, \alpha_j)_\tau := a$ (R. Wilson's reduced inner product)

- Like the other exceptional geometries, E_8 is actually hidden within 3D geometry!





New, explicit connections



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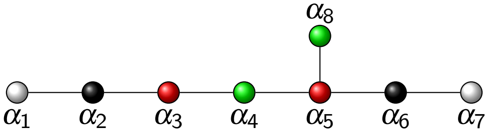
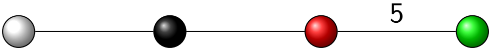
Projection and Diagram Foldings



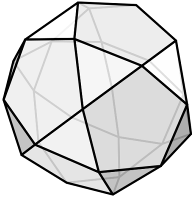
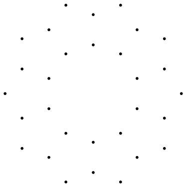
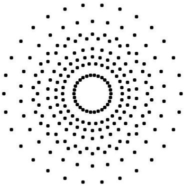


$$s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}, \; s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}, \; s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}, \; s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4$$

- E_8 has a H_4 subgroup of rotations via a 'partial folding'
- Can project 240 E_8 roots to $H_4 + \tau H_4$ – essentially the reverse of the previous construction!
- Coxeter element & number of E_8 and H_4 are the same

The Coxeter Plane

- Can show every (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by projecting their root system onto the Coxeter plane

Coxeter Elements, Degrees and Exponents

- Like the symmetric group, Coxeter groups can have invariant polynomials. Their degrees d are important invariants/group characteristics.
- Turns out that actually degrees d are intimately related to so-called exponents m $\boxed{m = d - 1}$.

Coxeter Elements, Degrees and Exponents

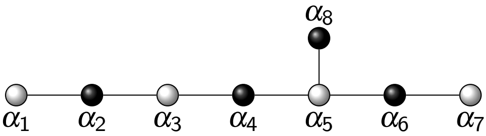
- A Coxeter Element is any combination of all the simple reflections $\boxed{w = s_1 \dots s_n}$, i.e. in Clifford algebra it is encoded by the versor $W = \alpha_1 \dots \alpha_n$ acting as $\boxed{v \rightarrow wv = \pm \tilde{W} v W}$. All such elements are conjugate and thus their order is invariant and called the Coxeter number h.
- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where m are called exponents:
  $\boxed{wx = \exp(2\pi mi/h)x}$
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure (or in fact, there are different ones).

Coxeter Elements, Degrees and Exponents

- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where m are called exponents
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again (the unfortunate standard procedure in many situations) – without any insight into the complex structure(s)
- In particular, 1 and $h-1$ are always exponents
- Turns out that actually exponents and degrees are intimately related ($\boxed{m = d - 1}$). The construction is slightly roundabout but uniform, and uses the Coxeter plane.

The Coxeter Plane

- In particular, can show every (for our purposes) Coxeter group has a Coxeter plane
- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of mutually commuting generators

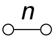


The Coxeter Plane

- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of orthogonal = mutually commuting generators but anticommuting root vectors α_w and α_b (duals ω)
- Cartan matrices are positive definite, and thus have a Perron-Frobenius (all positive) eigenvector λ_i.
- Take linear combinations of components of this eigenvector as coefficients of two vectors from the orthogonal sets $v_w = \sum \lambda_w \omega_w$ and $v_b = \sum \lambda_b \omega_b$
- Their outer product/Coxeter plane bivector $\boxed{B_C = v_b \wedge v_w}$ describes an invariant plane where w acts by rotation by $2\pi/h$.

Clifford Algebra and the Coxeter Plane – 2D case

$I_2(n)$



- For $I_2(n)$ take $\boxed{\alpha_1 = e_1, \ \alpha_2 = -\cos\frac{\pi}{n}e_1 + \sin\frac{\pi}{n}e_2}$
- So Coxeter versor is just

$$\boxed{W = \alpha_1\alpha_2 = -\cos\frac{\pi}{n} + \sin\frac{\pi}{n}e_1e_2 = -\exp\left(-\frac{\pi I}{n}\right)}$$

- In Clifford algebra it is therefore immediately obvious that the action of the $I_2(n)$ Coxeter element is described by a versor (here a rotor/spinor) that encodes rotations in the e_1e_2-Coxeter-plane and yields $h = n$ since trivially $W^n = (-1)^{n+1}$ yielding $\boxed{w^n = 1}$ via $wv = \tilde{W}vW$.

Clifford Algebra and the Coxeter Plane – 2D case

- So Coxeter versor is just $\boxed{W = -\exp\left(-\frac{\pi I}{n}\right)}$
- $\boxed{I = e_1 e_2}$ anticommutes with both e_1 and e_2 such that sandwiching formula becomes
  $$\boxed{v \rightarrow wv = \tilde{W} v W = \tilde{W}^2 v = \exp\left(\pm\frac{2\pi I}{n}\right) v}$$ immediately yielding the standard result for the complex eigenvalues in real Clifford algebra without any need for artificial complexification
- The Coxeter plane bivector $B_C = e_1 e_2 = I$ gives the complex structure
- The Coxeter plane bivector B_C is invariant under the Coxeter versor $\tilde{W} B_C W = \pm B_C$.

Clifford Algebra and the Coxeter Plane – 3D case

- In 3D, A_3, B_3, H_3 have $\{1,2,3\}$, $\{1,3,5\}$ and $\{1,5,9\}$
- Coxeter element is product of a spinor in the Coxeter plane with the same complex structure as before, and a reflection perpendicular to the plane
- So in 3D still completely determined by the plane
- 1 and $h-1$ are rotations in Coxeter plane
- $h/2$ is the reflection (for v in the normal direction)

$$\boxed{wv = \tilde{W}^2 = \exp(\pm\frac{2\pi I}{h}\frac{h}{2}) = \exp(\pm\pi I)v = -v}$$

Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can factorise W into orthogonal (commuting/anticommuting) components $\boxed{W = \alpha_1 \dots \alpha_n = W_1 \dots W_n}$ with $\boxed{W_i = \exp(\pi m_i I_i / h)}$
- Here, I_i is a bivector describing a plane with $\boxed{I_i^2 = -1}$
- For v orthogonal to the plane descrbed by I_i we have $\boxed{v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i W_i v = v}$ so cancels out
- For v in the plane we have $\boxed{v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i / h) v}$
- Thus if we decompose W into orthogonal eigenspaces, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

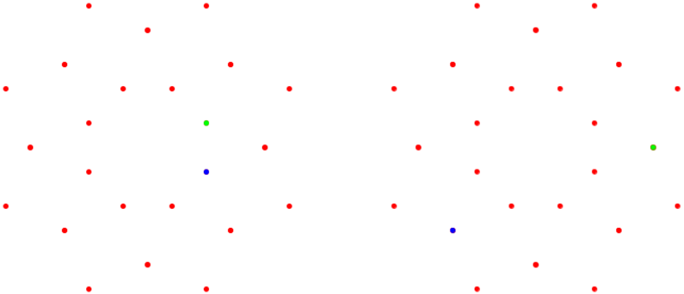
Clifford algebra: no need for complexification

- For v in the plane we have
  $$v \to \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i/h) v$$
- So complex eigenvalue equation arises geometrically without any need for complexification
- Different complex structures immediately give different eigenplanes
- Eigenvalues/angles/exponents given from just factorising $W = \alpha_1 \dots \alpha_n$
- E.g. B_4 has exponents $1, 3, 5, 7$ and $W = \exp\left(\frac{\pi}{8} I_1\right) \exp\left(\frac{3\pi}{8} I_2\right)$
- Here we have been looking for orthogonal eigenspaces, so innocuous – different complex structures commute
- But not in general – naive complexification can be misleading

4D case: B_4

- E.g. B_4 has exponents $1, 3, 5, 7$
- Coxeter versor decomposes into orthogonal components

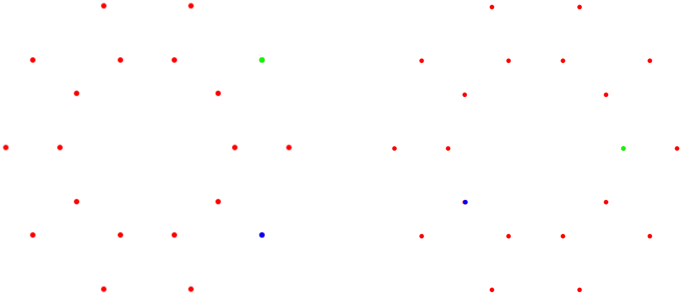
$$W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{8}B_C\right)\exp\left(\frac{3\pi}{8}IB_C\right)$$



4D case: A_4

- E.g. A_4 has exponents $1, 2, 3, 4$
- Coxeter versor decomposes into orthogonal components

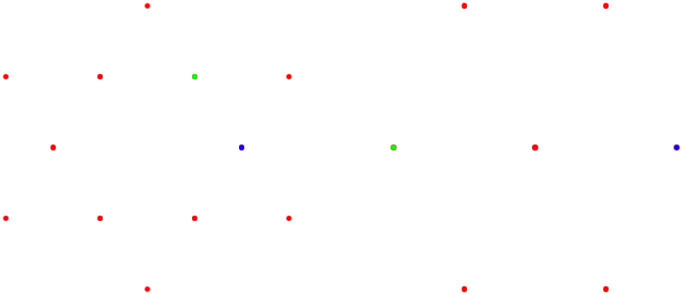
$$W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{5}B_C\right)\exp\left(\frac{2\pi}{5}IB_C\right)$$



4D case: D_4

- E.g. D_4 has exponents $1, 3, 3, 5$
- Coxeter versor decomposes into orthogonal components

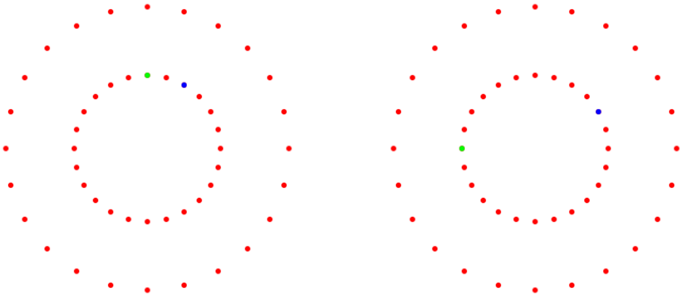
$$W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{6}B_C\right)\exp\left(\frac{3\pi}{6}IB_C\right)$$



4D case: F_4

- E.g. F_4 has exponents $1, 5, 7, 11$
- Coxeter versor decomposes into orthogonal components

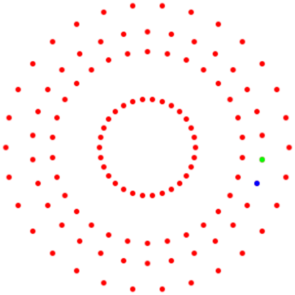
$$W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{12}B_C\right)\exp\left(\frac{5\pi}{12}IB_C\right)$$

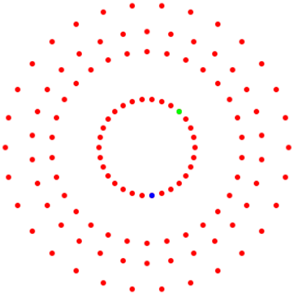


4D case: H_4

- E.g. H_4 has exponents $1, 11, 19, 29$
- Coxeter versor decomposes into orthogonal components

$$W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{11\pi}{30}IB_C\right)$$





Clifford Algebra and the Coxeter Plane – 4D case summary

rank 4	exponents	W-factorisation
A_4	$1,2,3,4$	$W = \exp\left(\frac{\pi}{5}B_C\right)\exp\left(\frac{2\pi}{5}IB_C\right)$
B_4	$1,3,5,7$	$W = \exp\left(\frac{\pi}{8}B_C\right)\exp\left(\frac{3\pi}{8}IB_C\right)$
D_4	$1,3,3,5$	$W = \exp\left(\frac{\pi}{6}B_C\right)\exp\left(\frac{\pi}{2}IB_C\right)$
F_4	$1,5,7,11$	$W = \exp\left(\frac{\pi}{12}B_C\right)\exp\left(\frac{5\pi}{12}IB_C\right)$
H_4	$1,11,19,29$	$W = \exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{11\pi}{30}IB_C\right)$

Actually, in 2, 3 and 4 dimensions it couldn't really be any other way

Clifford Algebra and the Coxeter Plane – D_6

- For D_6 one has exponents $\boxed{1, 3, 5, 5, 7, 9}$
- Coxeter versor decomposes into orthogonal bits as

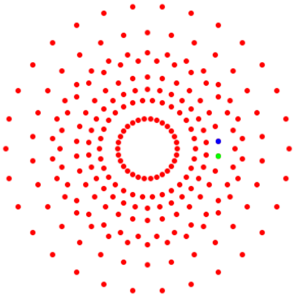
$$\boxed{W = \frac{1}{\sqrt{5}}(e_1 + e_2 + e_3 - e_4 - e_5)e_6 \exp\left(\frac{\pi}{10}B_C\right)\exp\left(\frac{3\pi}{10}B_2\right)}$$

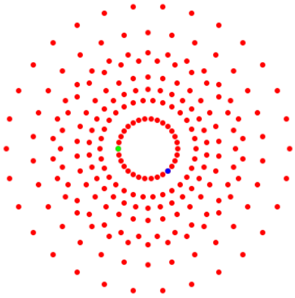
- Now bivector exponentials correspond to rotations in orthogonal planes
- Vector factors correspond to reflections
- For odd n, there is always one such vector factor in D_n, and for even n there are two

8D case: E_8

- E.g. H_4 has exponents $1, 11, 19, 29$, E_8 has $1, 7, 11, 13, 17, 19, 23, 29$
- Coxeter versor decomposes into orthogonal components

$$W = \alpha_1 \dots \alpha_8 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{7\pi}{30} B_2\right) \exp\left(\frac{11\pi}{30} B_3\right) \exp\left(\frac{13\pi}{30} B_4\right)$$

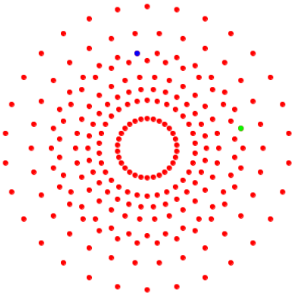


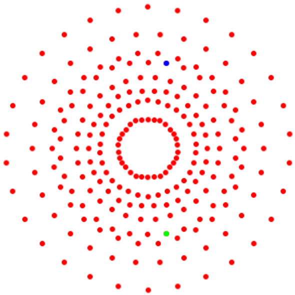


8D case: E_8

- E.g. H_4 has exponents $1, 11, 19, 29$, E_8 has $1, 7, 11, 13, 17, 19, 23, 29$
- Coxeter versor decomposes into orthogonal components

$$W = \alpha_1 \dots \alpha_8 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{7\pi}{30} B_2\right) \exp\left(\frac{11\pi}{30} B_3\right) \exp\left(\frac{13\pi}{30} B_4\right)$$





Imaginary differences – different imaginaries

So what has been gained by this Clifford view?

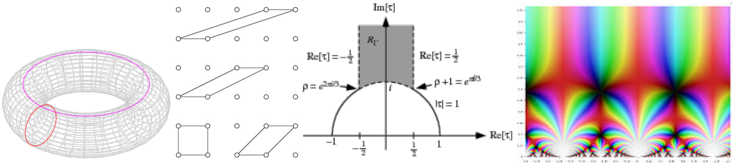
- There are different entities that serve as unit imaginaries
- They have a geometric interpretation as an eigenplane of the Coxeter element
- These don't need to commute with everything like i (though they do here – at least anticommute. But that is because we looked for orthogonal decompositions)
- But see that in general naive complexification can be a dangerous thing to do – unnecessary, issues of commutativity, confusing different imaginaries etc

Conclusions

- **All** exceptional geometries arise in 3D, root systems giving rise to Lie groups/algebras etc
- Completely novel spinorial way of viewing the geometries as 3D phenomena – implications for HEP etc?
- More natural point of view, explaining existence and perhaps automorphism groups
- Unclear how one would see this in a matrix framework – might require Clifford point of view
- New view of Coxeter degrees and exponents with geometric interpretation of imaginaries
- A unified framework for doing group and representation theory: polyhedral, orthogonal, conformal, modular (Moonshine) etc

Thank you!

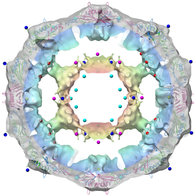
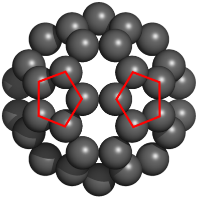
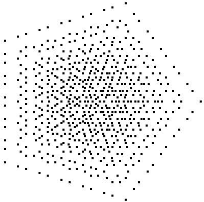
Modular group



- Modular group: interested in modular forms for applications in Moonshine/string theory: Monster 196883, Klein j 196884
- Modular generators: $T : \tau \to \tau + 1$, $S : \tau \to -1/\tau$
- $\boxed{\langle S, T | S^2 = I, (ST)^3 = I \rangle}$
- CGA: $T_X = 1 + \frac{ne_1}{2}$ and $S_X = e_1 e$
- $(S_X T_X)^3 = -1$ and $S_X^2 = 1$

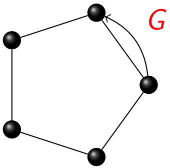
Motivation: Viruses

- Geometry of polyhedra described by Coxeter groups
- Viruses have to be 'economical' with their genes
- Encode structure modulo symmetry
- Largest discrete symmetry of space is the icosahedral group
- Many other 'maximally symmetric' objects in nature are also icosahedral: Fullerenes & Quasicrystals
- But: viruses are not just polyhedral – they have radial structure. Affine extensions give translations

Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon



Affine extensions of non-crystallographic root systems

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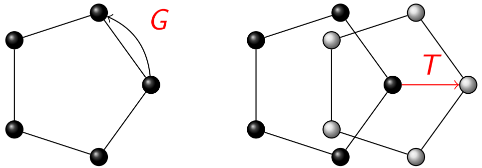

Affine extensions of non-crystallographic root systems

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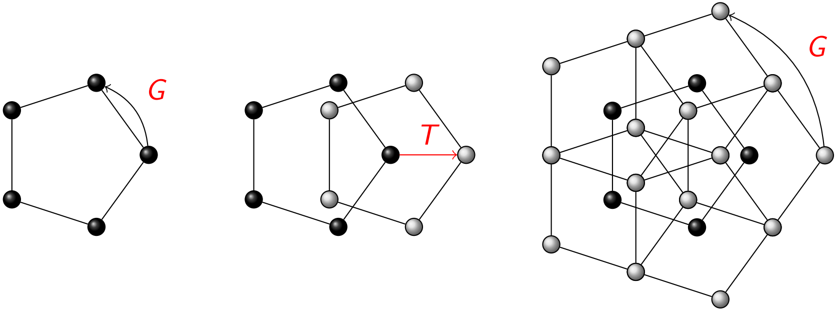
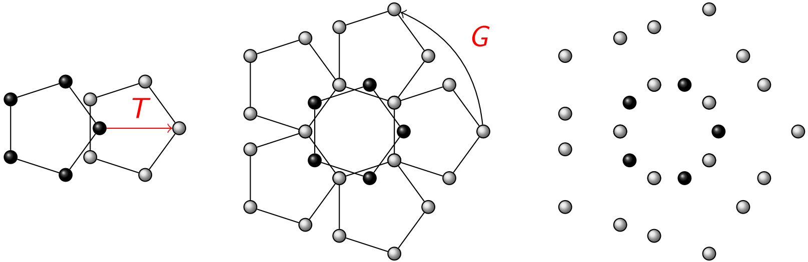


A random translation would give 5 secondary pentagons, i.e. 25 points. Here we have degeneracies due to 'coinciding points'.

Affine extensions of non-crystallographic root systems

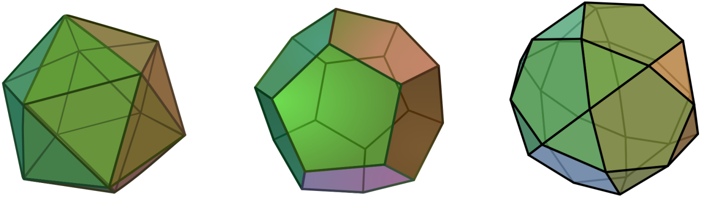
Translation of length $\tau = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$ (golden ratio)



Looks like a virus or carbon onion

Extend icosahedral group with distinguished translations

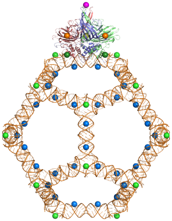
- Radial layers are simultaneously constrained by affine symmetry
- Works very well in practice: finite library of blueprints
- Select blueprint from the outer shape (capsid)
- Can predict inner structure (nucleic acid distribution) of the virus from the point array

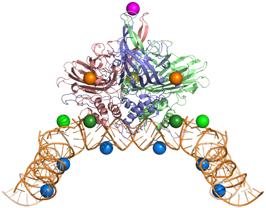


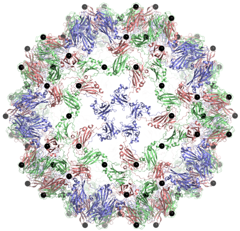
Affine extensions of the icosahedral group (giving translations) and their classification.

Use in Mathematical Virology

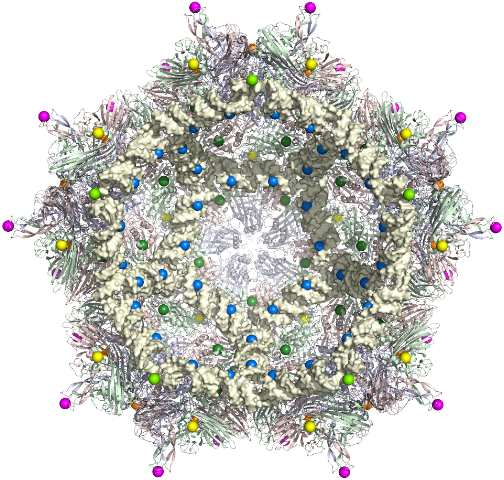
- Suffice to say point arrays work very exceedingly well in practice. Two papers on the mathematical (Coxeter) aspects.
- Implemented computational problem in Clifford – some very interesting mathematics comes out as well (see later).





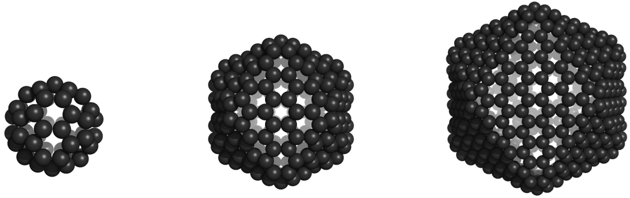


Use in Mathematical Virology



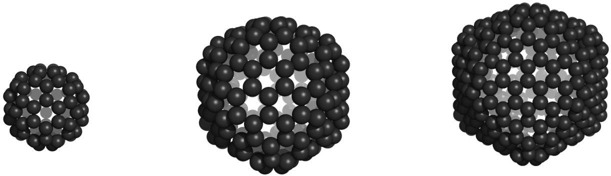
Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{60} - C_{240} - C_{540}$)



Extension to fullerenes: carbon onions

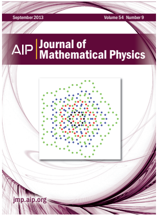
- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions ($C_{80} - C_{180} - C_{320}$)

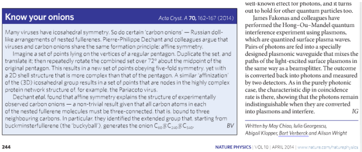


References

- Novel Kac-Moody-type affine extensions of non-crystallographic Coxeter groups with Twarock/Bœhm J. Phys. A: Math. Theor. 45 285202 (2012)
- Affine extensions of non-crystallographic Coxeter groups induced by projection with Twarock/Bœhm Journal of Mathematical Physics 54 093508 (2013), Cover article September
- Viruses and Fullerenes – Symmetry as a Common Thread? with Twarock/Wardman/Keef March Cover Acta Crystallographica A 70 (2). pp. 162-167 (2014), and Nature Physics Research Highlight

Applications of affine extensions of non-crystallographic root systems





Know your onions *Acta Cryst. A* **70,** 162–167 (2014)

Many viruses have icosahedral symmetry. So do certain 'carbon onions' — Russian doll-like arrangements of nested fullerenes. Pierre-Philippe Dechant and colleagues argue that viruses and carbon onions share the same formation principle: affine symmetry.

Imagine a set of points lying on the vertices of a regular pentagon. Duplicate the set, and translate it; then repeatedly rotate the combined set over 72° about the midpoint of the original pentagon. This results in a new set of points obeying five-fold symmetry, yet with a 2D shell structure that is more complex than that of the pentagon. A similar 'affinization' of the (3D) icosahedral group results in a set of points that are nodes in the highly complex protein network structure of, for example, the Pariacoto virus.

Dechant *et al.* found that affine symmetry explains the structure of experimentally observed carbon onions — a non-trivial result given that all carbon atoms in each of the nested fullerene molecules must be three-connected, that is, bound to three neighbouring carbons. In particular, they identified the extended group that, starting from buckminsterfullerene (the 'buckyball'), generates the onion $C_{60}@C_{240}@C_{540}$. *BV*

well-known effect for photons, and it turns out to hold for other quantum particles too.

James Fakonas and colleagues have performed the Hong–Ou–Mandel quantum interference experiment using plasmons, which are quantized surface plasma waves. Pairs of photons are fed into a specially designed plasmonic waveguide that mixes the paths of the light-excited surface plasmons in the same way as a beamsplitter. The outcome is converted back into photons and measured by two detectors. As in the purely photonic case, the characteristic dip in coincidence rate is there, showing that the photons remain indistinguishable when they are converted into plasmons and interfere. *IG*

Written by May Chiao, Iulia Georgescu, Abigail Klopper, Bart Verberck and Alison Wright

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There are interesting applications to quasicrystals, viruses or carbon onions, but here concentrate on the mathematical aspects

Quaternions and Clifford Algebra

- The unit spinors $\{1; Ie_1; Ie_2; Ie_3\}$ of $\mathrm{Cl}(3)$ are isomorphic to the quaternion algebra $\mathbb{H}$ (up to sign)
- The 3D Hodge dual of a vector is a pure bivector which corresponds to a pure quaternion, and their products are identical (up to sign)

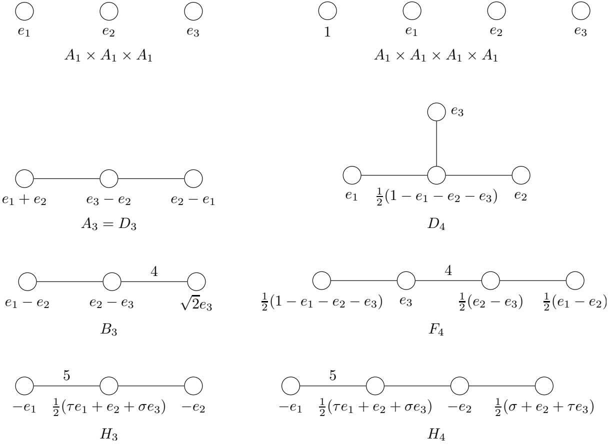
Discrete Quaternion groups

- The 8 quaternions of the form $(\pm 1, 0, 0, 0)$ and permutations are called the Lipschitz units, and form a realisation of the quaternion group in 8 elements.
- The 8 Lipschitz units together with $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ are called the Hurwitz units, and realise the binary tetrahedral group of order 24. Together with the 24 'dual' quaternions of the form $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$, they form a group isomorphic to the binary octahedral group of order 48.
- The 24 Hurwitz units together with the 96 unit quaternions of the form $(0, \pm\tau, \pm 1, \pm\sigma)$ and even permutations, are called the Icosians. The icosian group is isomorphic to the binary icosahedral group with 120 elements.

Quaternionic representations of 3D and 4D Coxeter groups

- Groups E_8, D_4, F_4 and H_4 have representations in terms of quaternions
- Extensively used in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why
- e.g. H_4 consists of 120 elements of the form $(\pm 1, 0, 0, 0)$, $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and $(0, \pm\tau, \pm 1, \pm\sigma)$
- Seen as remarkable that the subset of the 30 pure quaternions is a realisation of H_3 (a sub-root system)
- Similarly, A_3, B_3, $A_1 \times A_1 \times A_1$ have representations in terms of pure quaternions
- Will see there is a much simpler geometric explanation

Quaternionic representations used in the literature



Demystifying Quaternionic Representations

- 3D: Pure quaternions = Hodge dualised (pseudoscalar) root vectors
- In fact, they are the simple roots of the Coxeter groups
- 4D: Quaternions = disguised spinors – but those of the 3D Coxeter group i.e. the binary polyhedral groups!
- This relation between 3D and 4D via the geometric product does not seem to be known
- Quaternion multiplication = ordinary Clifford reflections and rotations

Demystifying Quaternionic Representations

- Pure quaternion subset of 4D groups only gives 3D group if the 3D group contains the inversion/pseudoscalar I
- e.g. does not work for the tetrahedral group A_3, but $A_3 \to D_4$ induction still works, with the central node essentially 'spinorial'
- In fact, it goes the other way around: the 3D groups induce the 4D groups via spinors
- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as $R_1 = \alpha_1\alpha_2$ and $R_2 = \alpha_2\alpha_3$
- Can see these are 'spinor generators' and how they don't really contain any more information/roots than the rank-3 groups alone

Quaternions vs Clifford versors

- Sandwiching is often seen as particularly nice feature of the quaternions giving rotations
- This is actually a general feature of Clifford algebras/versors in any dimension; the isomorphism to the quaternions is accidental to 3D
- However, the root system construction does not necessarily generalise
- 2D generalisation merely gives that $I_2(n)$ is self-dual
- Octonionic generalisation just induces two copies of the above 4D root systems, e.g. $A_3 \to D_4 \oplus D_4$